Homework Set 4: Exercises on Linear Maps

Directions: Please work on all of the following problems! Hand in the Calculational Problems 1 and 2, and the Proof-Writing Problems 6 and 7 at the **beginning** of lecture on February 2, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1. Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
   (a) Show that T is linear.
   (b) Show that T is surjective.
   (c) Find $\dim \operatorname{null} T$.
   (d) Find the matrix for T with respect to the canonical basis of $\mathbb{R}^2$.
   (e) Show that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + y, x + 1)$ is not linear.

2. Consider the complex vector spaces $\mathbb{C}^2$ and $\mathbb{C}^3$ with their canonical bases. Let $S : \mathbb{C}^3 \to \mathbb{C}^2$ be defined by the matrix

$$M(S) = A = \begin{pmatrix} i & 1 & 1 \\ 2i & -1 & -1 \end{pmatrix}.$$

   Find a basis for $\operatorname{null} S$.

3. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ having the property that

$$\forall\, a \in \mathbb{R}, \forall\, v \in \mathbb{R}, f(av) = af(v)$$

   but such that f is not a linear map.

4. Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V. Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

5. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is injective. Given a linearly independent list $(v_1, \ldots, v_n)$ of vectors in V, prove that the list $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

6. Let U, V, and W be vector spaces over $\mathbb{F}$, and suppose that the linear maps $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$ are both injective. Prove that the composition map $T \circ S$ is also injective.

7. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, \ldots, v_n)$ for V, prove that $\operatorname{span}(T(v_1), \ldots, T(v_n)) = W$.

8. Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional. Given $T \in \mathcal{L}(V, W)$, prove that there is a subspace U of V such that

$$U \cap \operatorname{null}(T) = \{0\} \quad \text{and} \quad \operatorname{range}(T) = \{T(u) \mid u \in U\}.$$

9. Show that the linear map $T : \mathbb{F}^4 \to \mathbb{F}^2$ is surjective if

$$\operatorname{null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2, x_3 = 7x_4\}.$$

10. Show that no linear map $T : \mathbb{F}^5 \to \mathbb{F}^2$ can have as its null space the set

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_1 = 3x_2, x_3 = x_4 = x_5\}.$$

11. Let V be a vector spaces over $\mathbb{F}$, and suppose that there is a linear map $T \in \mathcal{L}(V, V)$ such that both $\operatorname{null}(T)$ and $\operatorname{range}(T)$ are finite-dimensional subspaces of V. Prove that V must also be finite-dimensional.