Homework Set 9: Exercises on Orthogonality and Diagonalization

Directions: Please work on all of the problems and submit your solutions to the Calculational Problems 1 and 2, and Proof-Writing Problems 3 and 6 at the **beginning** of lecture on March 9, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. We also use $\langle \cdot , \cdot \rangle$ to denote an arbitrary inner product and $\| \cdot \|$ to denote its associated norm. The term "o.n." means *orthonormal*. A^* denotes the conjugate transpose of a matrix $A \in \mathbb{C}^{n \times n}$.

1. Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$, where

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1),\ f_2 = \frac{1}{\sqrt{6}}(1,-2,1),\ f_3 = \frac{1}{\sqrt{2}}(1,0,-1)\,.$$

   (a) Find the matrix, S, of the change of basis transformation such that

$$[v]_f = S[v]_e, \quad \text{for all } v \in \mathbb{R}^3\,,$$

   where $[v]_b$ denotes the column vector with the coordinates of the vector v in the basis b.

   (b) Find the canonical matrix, A, of the linear map $T \in \mathcal{L}(\mathbb{R}^3)$ with eigenvectors f_1, f_2, f_3 and eigenvalues $1, 1/2, -1/2$, respectively.

2. Let $v \in \mathbb{C}^4$ be the vector given by $v = (1, i, -1, -i)$. Find the matrix (with respect to the canonical basis on $\mathbb{C}^4$) of the orthogonal projection $P \in \mathcal{L}(\mathbb{C}^4)$ such that

$$\mathrm{null}(P) = \{v\}^{\perp}\,.$$

3. Let U be the subspace of $\mathbb{R}^3$ that coincides with the plane through the origin that is perpendicular to the vector $n = (1,1,1) \in \mathbb{R}^3$.

   (a) Find an o.n. basis for U.

   (b) Find the matrix (with respect to the canonical basis on $\mathbb{R}^3$) of the orthogonal projection $P \in \mathcal{L}(\mathbb{R}^3)$ onto U, i.e., such that $\mathrm{range}(P) = U$.

4. For each of the following matrices, verify that A is Hermitian by showing that $A = A^*$, find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix, and compute $\exp(A)$.

(a) $A = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 6 & 2+2i \\ 2-2i & 4 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix}$ (e) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$ (f) $A = \begin{bmatrix} 2 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 2 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$

5. For each of the following matrices, either find a matrix P (not necessarily unitary) such that $P^{-1}AP$ is a diagonal matrix, or show why no such matrix exists.

(a) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ (e) $A = \begin{bmatrix} -i & 1 & 1 \\ -i & 1 & 1 \\ -i & 1 & 1 \end{bmatrix}$ (f) $A = \begin{bmatrix} 0 & 0 & i \\ 4 & 0 & i \\ 0 & 0 & i \end{bmatrix}$

6. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and suppose that $T \in \mathcal{L}(V)$ has the property that $T^* = -T$. (We call T a *skew Hermitian* operator on V.)

   (a) Prove that the operator $iT \in \mathcal{L}(V)$ defined by $(iT)(v) = i(T(v))$, for each $v \in V$, is Hermitian.

   (b) Prove that the canonical matrix for T can be unitarily diagonalized.

   (c) Prove that T has purely imaginary eigenvalues.