What Is Linear Algebra?

Isaiah Lankham, Bruno Nachtergaele, Anne Schilling
(January 7, 2007)

1 Introduction

Linear Algebra is the branch of mathematics aimed at solving systems of linear equations with a finite number of unknowns. In particular one would like to obtain answers to the following questions:

- **Characterization of solutions:** Are there solutions to a given system of linear equations? How many solutions are there?
- **Finding solutions:** How does the solution set look like? What are the solutions?

Linear Algebra is a systematic theory regarding the solutions of systems of linear equations.

Example 1.1. Let us take the system of two linear equations in two unknowns x_1 and x_2

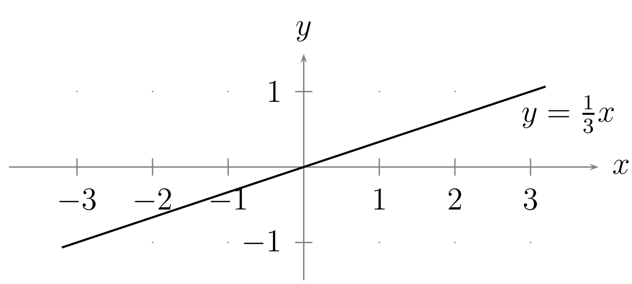
$$\begin{aligned} 2x_1 + x_2 &= 0, \\ x_1 - x_2 &= 1. \end{aligned}$$

It has a **unique** solution for $x_1, x_2 \in \mathbb{R}$, namely $x_1 = \frac{1}{3}$ and $x_2 = -\frac{2}{3}$.

Example 1.2. Let us take the system of one linear equation in two unknowns x_1 and x_2

$$x_1 - 3x_2 = 0.$$

In this case there are **infinitely many** solutions given by the set $\{x_2 = \frac{1}{3}x_1 \mid x_1 \in \mathbb{R}\}$. You can think of this solution set as a line in 2-space:



2 Systems of linear equations

2.1 Linear equations

Before going on, let us be a bit more precise about what we mean by **a system of linear equations**. A **function** f is a map

$$f : X \to Y \tag{1}$$

from a set X to a set Y. The set X is called the **domain** of the function and the set Y is called the **target space** or **codomain** of the function. An **equation** is

$$f(x) = y, \tag{2}$$

where $x \in X$ and $y \in Y$.

Example 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x$. Then $f(x) = x^3 - x = 1$ is an equation. The domain and target space are both the set of real numbers $\mathbb{R}$ in this case.

(If you are not familiar with the notion of sets and functions, there will be a set of additional lecture notes devoted entirely to this topic to be discussed in the first discussion session).

In this setting a system of equations is just another kind of equation.

Example 2.2. Let $X = Y = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the Cartesian product of the set of real numbers. Then define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(x_1, x_2) = (2x_1 + x_2, x_1 - x_2) \tag{3}$$

and $y = (0, 1)$. Then the equation $f(x) = y$, where $x = (x_1, x_2) \in \mathbb{R}^2$ describes the system of linear equations of Example 1.1.

The next question is, what is a linear equation? A **linear equation** is defined by a linear function f defined on a linear space, also known as a **vector space**. We will elaborate on this in future lectures, but let us demonstrate the main feature of a vector space in terms of the example $\mathbb{R}^2$. Take $x = (x_1, x_2), z = (z_1, z_2) \in \mathbb{R}^2$. There are two operations defined on $\mathbb{R}^2$, namely addition and scalar multiplication

$$x + z := (x_1 + z_1, x_2 + z_2) \qquad \text{(vector addition)}, \tag{4}$$

$$cx := (cx_1, cx_2) \qquad \text{(scalar multiplication)}. \tag{5}$$

A linear function is a function f such that

$$f(cx) = cf(x) \tag{6}$$
$$f(x+z) = f(x) + f(z). \tag{7}$$

Please check for yourself that the function f of Example 2.2 has these properties.

2.2 Non-linear equations

(Systems of) Linear equations are a very important class of (systems of) equations. For example, we will learn in this class how to solve systems of linear equations in general. Non-linear equations are much harder to solve. An example of a quadratic equation is

$$x^2 + x - 2 = 0. \tag{8}$$

It has two solutions $x = -2$ and $x = 1$. What about the equation

$$x^2 + x + 2 = 0? \tag{9}$$

This equation does not have any solution in the set of real numbers. Next lecture we will discuss complex numbers in more detail.

(Recall that the quadratic equation $x^2 + bx + c = 0$ has the two solutions

$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$.)

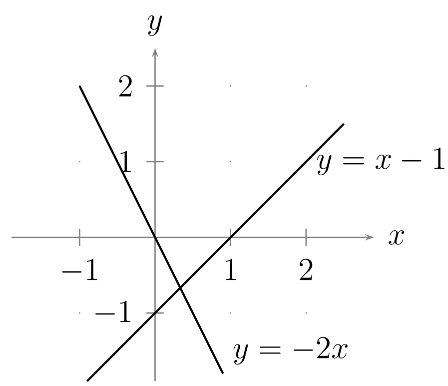
2.3 Linear transformations

The set $\mathbb{R}^2$ can be viewed as the plane. In this context linear functions $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be interpreted geometrically as "motions" in the plane and are called **linear transformations**.

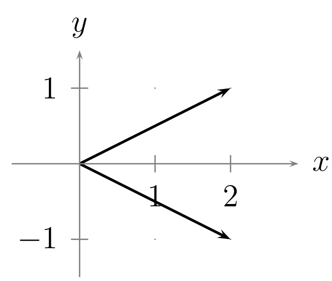
Example 2.3. Recall the two linear equations of Example 1.1

$$2x_1 + x_2 = 0,$$
$$x_1 - x_2 = 1.$$

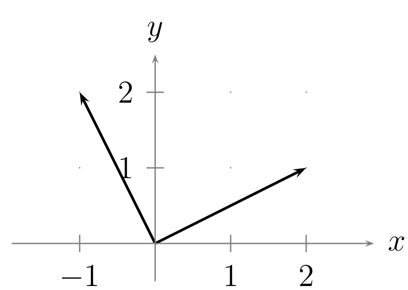
Each equation can be interpreted as a straight line in the plane. The solutions (x_1, x_2) to both equations are given by the set of points that lie on both lines.



Example 2.4. The linear map $f(x_1, x_2) = (x_1, -x_2)$ describes the reflection across the x-axis.



Example 2.5. The linear map $f(x_1, x_2) = (-x_2, x_1)$ describes the rotation by 90^0 counter-clockwise.

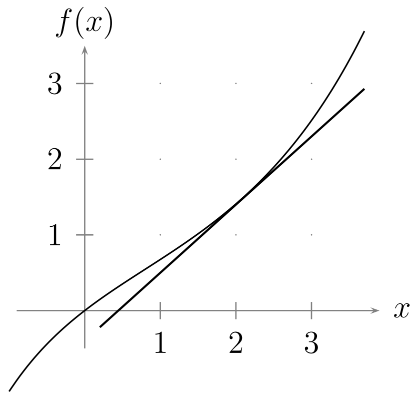


Example 2.6. For an angle $\theta \in [0, 2\pi)$, find the linear map f_θ which describes the rotation by the angle θ in the counterclockwise direction.
Hint: For a given angle θ find a, b, c, d such that $f_\theta(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$.

2.4 Applications of linear equations

Linear equations pop up in many different contexts. For example you can view the derivative df/dx of a function $f : \mathbb{R} \to \mathbb{R}$ as a linear approximation of f. This becomes apparent when you look at the Taylor series of the function $f(x)$ around the point a (see MAT 21C)

$$f(x) = f(a) + df/dx(a)(x - a) + \cdots \tag{10}$$



If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function of more than one variable, one can still view the derivative of f as the linear approximation of f; see MAT 21D.

What about infinitely many variables $x_1, x_2, \ldots$? In this case the system of equations has the form

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots &= y_1 \\ b_1x_1 + b_2x_2 + \cdots &= y_2 \\ &\cdots \end{aligned}$$

Hence the sums are infinite, so that one is dealing with series and the question of convergence arises. Convergence depends on $x = (x_1, x_2, \ldots)$ which is the variable we want to solve for. These questions will not arise in this course, as we are only interested in finite systems of linear equations in a finite number of variables. Other courses where these questions pop up are

- Differential Equations MAT 22B, MAT 118AB;
- Fourier Analysis MAT 129;

- Analysis MAT 125AB, MAT 185AB, MAT 201ABC, MAT 202.

In MAT 150ABC and MAT 250ABC linear algebra arises in the study of symmetries, linear transformations or in Lie algebra theory.