Permutations and the Determinant

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1 Introduction

Given a positive integer $n \in \mathbb{Z}_+$, a *permutation* of an (ordered) list of n distinct objects is any reordering of this list. When describing the reorderings themselves, though, note that the nature of the objects involved is more or less irrelevant. E.g., we can imagine interchanging the second and third items in a list of five distinct objects, no matter what those items are, and this defines a particular permutation upon any list of five objects.

Since the nature of the objects being rearranged (i.e., permuted) is immaterial, it is common to use the integers $1, 2, \ldots, n$, as the standard list of n objects. Alternatively, one can also think of these integers as labels for the items in any list of n distinct elements.

2 Definition for and examples of permutations

Let $n \in \mathbb{Z}_+$ be a positive integer. Then, mathematically, we define a permutation as any invertible (a.k.a. bijective) transformation of the finite set $\{1, \ldots, n\}$ into itself.

Definition 2.1. A *permutation* π of n elements is a one-to-one and onto function having the set $\{1, 2, \ldots, n\}$ as both its domain and codomain.

In other words, a permutation is a function $\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ such that, for every integer $i \in \{1, \ldots, n\}$, there exists exactly one integer $j \in \{1, \ldots, n\}$ for which $\pi(j) = i$. We will usually denote permutations by Greek letters such as π (pi), σ (sigma), and τ (tau). The set of all permutations of n elements is denoted by $\mathcal{S}_n$ and is commonly called the *symmetric group of degree n*. (In particular, the set $\mathcal{S}_n$ forms a group under function composition as discussed in Section 3 below.)

Given a permutation $\pi \in \mathcal{S}_n$, there are several common notations used for specifying how π permutes the integers $1, 2, \ldots, n$. The important thing to keep in mind when working with any of these notations is that π is a function defined on the finite set $\{1, 2, \ldots, n\}$, with notation a convenient short-hand for keeping track of how π transforms this set.

Definition 2.2. Given a *permutation* $\pi \in \mathcal{S}_n$, denote $\pi_i = \pi(i)$ for each $i \in \{1, \ldots, n\}$. Then the *two-line notation* for π is given by the $2 \times n$ matrix

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}.$$

In other words, given a permutation $\pi \in \mathcal{S}_n$ and an integer $i \in \{1, \ldots, n\}$, we are denoting the image of i under π by π_i instead of using the more conventional function notation $\pi(i)$. Then, in order to specify the image of each integer $i \in \{1, \ldots, n\}$ under π, we list these images in a two-line array as shown above. (One can also use so-called *one-line notation* for π, which is given by simply ignoring the top row and writing $\pi = \pi_1 \pi_2 \cdots \pi_n$.)

It is important to note that, although we represent permutations as $2 \times n$ matrices, you *should not* think of permutations as linear transformations from an n-dimensional vector space to a two-dimensional vector space. Moreover, the composition operation on permutation that we describe in Section 3 below *does not* correspond to matrix multiplication. The use of matrix notation in denoting permutations is merely a matter of convenience.

Example 2.3. Suppose that we have a set of five distinct objects and that we wish to describe the permutation that places the first item into the second position, the second item into the fifth position, the third item into the first position, the fourth item into the third position, and the fifth item into the fourth position. Then, using the notation developed above, we have the permutation $\pi \in \mathcal{S}_5$ such that

$$\pi_1 = \pi(1) = 3, \quad \pi_2 = \pi(2) = 1, \quad \pi_3 = \pi(3) = 4, \quad \pi_4 = \pi(4) = 5, \quad \pi_5 = \pi(5) = 2.$$

Moreover, written in two-line notation,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}.$$

It is relatively straightforward to find the number of permutations of n elements, i.e., to determine cardinality of the set $\mathcal{S}_n$. To construct an arbitrary permutation of n elements, we can proceed as follows: First, choose an integer $i \in \{1, \ldots, n\}$ to put in the first position. Clearly, we have exactly n possible choices. Next, choose the element to go in the second position. Since we have already chosen one element from the set $\{1, \ldots, n\}$, there are now exactly $n-1$ remaining choices. Proceeding in this way, we have $n-2$ choices when choosing the third element from the set $\{1, \ldots, n\}$, then $n-3$ choices when choosing the fourth element, and so on until we are left with exactly one choice for the n^{th} element.

Theorem 2.4. *The number of elements in the symmetric group* $\mathcal{S}_n$ *is given by*

$$|\mathcal{S}_n| = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = n!$$

We conclude this section by describing the one permutation in $\mathcal{S}_1$, the two permutations in $\mathcal{S}_2$, and the six permutations in $\mathcal{S}_3$. For your own practice, you should (patiently) attempt to list the $4! = 24$ permutations in $\mathcal{S}_4$.

Example 2.5.

1. Given any positive integer $n \in \mathbb{Z}_+$, the identity function $\mathrm{id} : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ given by $\mathrm{id}(i) = i$, $\forall\ i \in \{1, \ldots, n\}$, is a permutation in $\mathcal{S}_n$. This function can be thought of as the trivial reordering that does not change the order at all, and so we call it the *trivial* or *identity* permutation.

2. If $n = 1$, then, by Theorem 2.4, $|\mathcal{S}_n| = 1! = 1$. Thus, $\mathcal{S}_1$ contains on the identity permutation.

3. If $n = 2$, then, by Theorem 2.4, $|\mathcal{S}_n| = 2! = 2 \cdot 1 = 2$. Thus, there is only one non-trivial permutation π in $\mathcal{S}_2$, namely the transformation interchanging the first and the second elements in a list. As a function, $\pi(1) = 2$ and $\pi(2) = 1$, and, in two-line notation,
$$\pi = \begin{pmatrix} 1 & 2 \\ \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

4. If $n = 3$, then, by Theorem 2.4, $|\mathcal{S}_n| = 3! = 3 \cdot 2 \cdot 1 = 6$. Thus, there are five non-trivial permutation in $\mathcal{S}_3$. Using two-line notation, we have that
$$\mathcal{S}_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

   Once more, you should always regard a permutation as being simultaneously a function and a reordering operation. E.g., the permutation
$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
   can be read as defining the reordering that, with respect to the original list, places the second element in the first position, the third element in the second position, and the first element in the third position. This permutation could equally well have been identified by describing its action on the (ordered) list of letters a, b, c. In other words,
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix},$$
   regardless of what the letters a, b, c might happen to represent. In particular, if we set $a = 2$, $b = 1$, and $c = 3$, then the above equally becomes
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

3 Composition of permutations

Let $n \in \mathbb{Z}_+$ be a positive integer and $\pi, \sigma \in \mathcal{S}_n$ be permutations. Then, since π and σ are both functions from the set $\{1, \ldots, n\}$ to itself, we can compose them to obtain a new function $\pi \circ \sigma$ (read *pi after sigma*) that takes on the values

$$(\pi \circ \sigma)(1) = \pi(\sigma(1)), \quad (\pi \circ \sigma)(1) = \pi(\sigma(2)), \quad \ldots \quad (\pi \circ \sigma)(n) = \pi(\sigma(n)).$$

In two-line notation, we can write $\pi \circ \sigma$ as

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(\sigma(1)) & \pi(\sigma(2)) & \cdots & \pi(\sigma(n)) \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_{\sigma(1)} & \pi_{\sigma(2)} & \cdots & \pi_{\sigma(n)} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_{\sigma_1} & \pi_{\sigma_2} & \cdots & \pi_{\sigma_n} \end{pmatrix}.$$

Example 3.1. From $\mathcal{S}_3$, suppose that we have the permutations π and σ given by

$$\pi(1) = 2,\ \pi(2) = 3,\ \pi(3) = 1 \ \text{ and } \ \sigma(1) = 1,\ \sigma(2) = 3,\ \sigma(3) = 2.$$

Then note that

$$\begin{aligned} (\pi \circ \sigma)(1) &= \pi(\sigma(1)) = \pi(1) = 2, \\ (\pi \circ \sigma)(2) &= \pi(\sigma(2)) = \pi(3) = 1, \\ (\pi \circ \sigma)(3) &= \pi(\sigma(3)) = \pi(2) = 3. \end{aligned}$$

In other words,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(3) & \pi(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Similar computations (which you should check for your own practice) yield compositions such as

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(2) & \sigma(3) & \sigma(1) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \mathrm{id}(2) & \mathrm{id}(3) & \mathrm{id}(1) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

In particular, note that the result of each composition above is a permutation, that composition is not a commutative operation, and that composition with id leaves a permutation unchanged. Moreover, since each permutation π is a bijection, one can always construct an inverse permutation π^{-1} such that $\pi \circ \pi^{-1} = \mathrm{id}$. E.g.,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \pi(3) & \pi(1) & \pi(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Theorem 3.2. *Let $n \in \mathbb{Z}_+$ be a positive integer. Then the set $\mathcal{S}_n$ has the following properties.*

1. *Given any two permutations $\pi, \sigma \in \mathcal{S}_n$, the composition $\pi \circ \sigma \in \mathcal{S}_n$.*
2. *(Associativity of Composition) Given any three permutations $\pi, \sigma, \tau \in \mathcal{S}_n$,*
$$(\pi \circ \sigma) \circ \tau = \pi \circ (\sigma \circ \tau).$$
3. *(Identity Element for Composition) Given any permutation $\pi \in \mathcal{S}_n$,*
$$\pi \circ id = id \circ \pi = \pi.$$
4. *(Inverse Elements for Composition) Given any permutation $\pi \in \mathcal{S}_n$, there exists a unique permutation $\pi^{-1} \in \mathcal{S}_n$ such that*
$$\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = id.$$

In other words, the set $\mathcal{S}_n$ forms a group under composition.

Note that the composition of permutations is not commutative in general. In particular, for $n \geq 3$, you can easily find examples of permutations π and σ such that $\pi \circ \sigma \neq \sigma \circ \pi$.

4 Inversions and the sign of a permutation

Let $n \in \mathbb{Z}_+$ be a positive integer. Then, given a permutation $\pi \in \mathcal{S}_n$, it is natural to ask how "out of order" π is in comparison to the identity permutation. One method for quantifying this is to count the number of so-called *inversion pairs* in π as these describe pairs of objects that are out of order relative to each other.

Definition 4.1. Let $\pi \in \mathcal{S}_n$ be a permutation. Then an *inversion pair* (i, j) of π is a pair of positive integers $i, j \in \{1, \ldots, n\}$ for which $i < j$ but $\pi(i) > \pi(j)$.

Note, in particular, that the components of an inversion pair are the *positions* where the two "out of order" elements occur.

Example 4.2. We classify all inversion pairs for elements in $\mathcal{S}_3$:

- $\mathrm{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ has no inversion pairs since no elements are "out of order".
- $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ has the single inversion pair $(2, 3)$ since $\pi(2) = 3 > 2 = \pi(3)$.

- $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ has the single inversion pair $(1, 2)$ since $\pi(1) = 2 > 1 = \pi(2)$.

- $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ has the two inversion pairs $(1, 3)$ and $(2, 3)$ since we have that both $\pi(1) = 2 > 1 = \pi(3)$ and $\pi(2) = 3 > 1 = \pi(3)$.

- $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ has the two inversion pairs $(1, 2)$ and $(1, 3)$ since we have that both $\pi(1) = 3 > 1 = \pi(2)$ and $\pi(1) = 3 > 2 = \pi(3)$.

- $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ has the three inversion pairs $(1, 2)$, $(1, 3)$, and $(2, 3)$, as you can check.

Example 4.3. As another example, for each $i, j \in \{1, \ldots, n\}$ with $i < j$, we define the *transposition* $t_{ij} \in \mathcal{S}_n$ by

$$t_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ 1 & 2 & \cdots & j & \cdots & i & \cdots & n \end{pmatrix}.$$

In other words, t_{ij} is the permutation that interchanges i and j while leaving all other integers fixed in place. One can check that the number of inversions pairs in t_{ij} is exactly $2(j-i)-1$. Thus, the number of inversions in a transposition is always odd. E.g.,

$$t_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

has inversion pairs $(1, 2)$, $(1, 3)$, and $(2, 3)$.

For the purposes of this course, the significance of inversion pairs is mainly due to the following fundamental definition.

Definition 4.4. Let $\pi \in \mathcal{S}_n$ be a permutation. Then the *sign* of π, denoted by $\mathrm{sign}(\pi)$ is defined by

$$\mathrm{sign}(\pi) = (-1)^{\#\text{ of inversion pairs in } \pi} = \begin{cases} +1, & \text{if the number of inversions in } \pi \text{ is even} \\ -1, & \text{if the number of inversions in } \pi \text{ is odd} \end{cases}.$$

Moreover, we call π an *even permutation* if $\mathrm{sign}(\pi) = +1$, and we call π an *odd permutation* if $\mathrm{sign}(\pi) = -1$.

Example 4.5. Based upon the computations in Example 4.2 above, we have that

$$\operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = \operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = \operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = +1$$

and that

$$\operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} = \operatorname{sign}\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = -1.$$

Similarly, from Example 4.3, it follows that any transposition is an odd permutation.

We summarize some of the most basic properties of the sign operation on the symmetric group in the following theorem.

Theorem 4.6. *Let $n \in \mathbb{Z}_+$ be a positive integer. Then,*

- *for $id \in \mathcal{S}_n$ the identity permutation,*

$$sign(id) = +1.$$

- *for $t_{ij} \in \mathcal{S}_n$ a transposition with $i, j \in \{1, \ldots, n\}$ and $i < j$,*

$$sign(t_{ij}) = -1. \tag{1}$$

- *given any two permutations $\pi, \sigma \in \mathcal{S}_n$,*

$$\begin{aligned} sign(\pi \circ \sigma) &= sign(\pi)\, sign(\sigma), &(2)\\ sign(\pi^{-1}) &= sign(\pi). &(3) \end{aligned}$$

- *the number of even permutations in $\mathcal{S}_n$, when $n \geq 2$, is exactly $\frac{1}{2}n!$.*
- *the set A_n of even permutations in $\mathcal{S}_n$ forms a group under composition.*

5 Summations indexed by the set of all permutations

Let $n \in \mathbb{Z}_+$ be a positive integer, and recall the following definition:

Definition 5.1. Given a square matrix $A = (a_{ij}) \in \mathbb{F}_{n\times n}$, the *determinant* of A is

$$\det(A) = \sum_{\pi \in \mathcal{S}_n} \operatorname{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}\,, \tag{4}$$

where the sum is over all permutations of n elements (i.e., over the symmetric group).

Example 5.2. Take the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

To calculate the determinant of A, let us first list again the two permutations in S_2

$$\mathrm{id} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The permutation id has sign 1 and the permutation σ has sign -1. Hence the determinant is given by

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

Were one to attempt to compute determinants directly using Equation (4), then one would need to sum up $n!$ terms, where each summand is itself a product of n factors. This is an incredibly inefficient method for finding determinants since $n!$ increases in size very rapidly as n increases. E.g., $10! = 3628800$. Thus, even if you could compute one summand per second without stopping, then it would still take you well over a month to compute the determinant of a 10×10 matrix using Equation (4). Fortunately, there are properties of the determinant (as summarized in Section 6 below) that can be used to greatly reduce the size of such computations. These properties of the determinant follow from general properties that hold for any summation taken over the symmetric group, which are in turn themselves based upon properties of permutations and the fact that addition and multiplication are commutative operations in a field $\mathbb{F}$ (which, as usual, we take to be either $\mathbb{R}$ or $\mathbb{C}$).

Let $T : \mathcal{S}_n \to V$ be a function defined on the symmetric group $\mathcal{S}_n$ that takes values in some vector space V. E.g., $T(\pi)$ could be the term corresponding to the permutation π in the sum that defines the determinant of A. Then, since the sum

$$\sum_{\pi \in \mathcal{S}_n} T(\pi)$$

is finite, we are free to reorder the summands. In other words, the sum is independent of the order in which the terms are added, and so we can permute the term order freely without affecting the value of the sum. Some commonly used reorderings of such sums are the following:

$$\begin{aligned} \sum_{\pi \in \mathcal{S}_n} T(\pi) &= \sum_{\pi \in \mathcal{S}_n} T(\sigma \circ \pi) && (5) \\ &= \sum_{\pi \in \mathcal{S}_n} T(\pi \circ \sigma) && (6) \\ &= \sum_{\pi \in \mathcal{S}_n} T(\pi^{-1}) && (7) \end{aligned}$$

where σ is a fixed permutation.

Equation (5) follows from the fact that, if π runs through each permutation in $\mathcal{S}_n$ exactly once, then $\sigma \circ \pi$ similarly runs through each permutation but in a potentially different orders. I.e., the action of σ upon something like Equation (4) is that σ merely permutes the permutations that index the terms. Put another way, there is a one-to-one correspondence between permutations in general and permutations composed with σ.

Similar reasoning holds for Equations (6) and (7).

6 Properties of the determinant

We summarize some of the most basic properties of the determinant below. The proof of the following theorem uses properties of permutations, properties of the sign function on permutations, and properties of sums over the symmetric group as discussed in Section 5 above. In thinking about these properties, it is useful to keep in mind that, using Equation (4), the determinant of an $n \times n$ matrix A is the sum over all possible ways of selecting n entries of A, where exactly one element is selected from each row and from each column of A.

Theorem 6.1 (Properties of the Determinant). *Let $n \in \mathbb{Z}_+$ be a positive integer, and suppose that $A = (a_{ij}) \in \mathbb{F}^{n\times n}$ is an $n \times n$ matrix. Then*

1. $\det(0_{n\times n}) = 0$ *and* $\det(I_n) = 1$*, where $0_{n\times n}$ denotes the $n \times n$ zero matrix and I_n denotes the $n \times n$ identity matrix.*

2. $\det(A^T) = \det(A)$*, where A^T denotes the transpose of A.*

3. *denoting by $A^{(\cdot,1)}, A^{(\cdot,2)}, \ldots, A^{(\cdot,n)} \in \mathbb{F}^n$ the columns of A,* $\det A$ *is a linear function of column $A^{(\cdot,i)}$, for each $i \in \{1, \ldots, n\}$. In other words, if we denote*

$$A = \left[A^{(\cdot,1)} \mid A^{(\cdot,2)} \mid \cdots \mid A^{(\cdot,n)}\right]$$

then, given any scalar $z \in \mathbb{F}$ and any vectors $a_1, a_2, \ldots, a_n, c, b \in \mathbb{F}^n$,

$$\begin{aligned} \det\left[a_1 \mid \cdots \mid a_{i-1} \mid z a_i \mid \cdots \mid a_n\right] &= z \det\left[a_1 \mid \cdots \mid a_{i-1} \mid a_i \mid \cdots \mid a_n\right], \\ \det\left[a_1 \mid \cdots \mid a_{i-1} \mid b + c \mid \cdots \mid a_n\right] &= \det\left[a_1 \mid \cdots \mid b \mid \cdots \mid a_n\right] + \det\left[a_1 \mid \cdots \mid c \mid \cdots \mid a_n\right]. \end{aligned}$$

4. $\det(A)$ *is an antisymmetric function of the columns of A. In other words, given any positive integers $1 \le i < j \le n$ and denoting $A = \left[A^{(\cdot,1)} \mid A^{(\cdot,2)} \mid \cdots \mid A^{(\cdot,n)}\right]$,*

$$\det(A) = -\det\left[A^{(\cdot,1)} \mid \cdots \mid A^{(\cdot,j)} \mid \cdots \mid A^{(\cdot,i)} \mid \cdots \mid A^{(\cdot,n)}\right].$$

5. *if A has two identical columns,* $\det(A) = 0$.

6. *if A has a column of zero's,* $\det(A) = 0$.
7. *Properties 3–6 also hold when rows are used in place of columns.*
8. *given any other matrix $B \in \mathbb{F}^{n\times n}$,*

$$\det(AB) = \det(A)\ \det(B).$$

9. *if A is either upper triangular or lower triangular,*

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Proof. First, note that Properties 1, 3, 6, and 9 follow directly from the sum given in Equation (4). Moreover, Property 5 follows directly from Property 4, and Property 7 follows directly from Property 2. Thus, we need only prove Properties 2, 4, and 8.

Proof of 2. Since the entries of A^T are obtained from those of A by interchanging the row and column indices, it follows that $\det(A^T)$ is given by

$$\det(A^T) = \sum_{\pi\,\in\,\mathcal{S}_n} \operatorname{sign}(\pi)\ a_{\pi(1),1}a_{\pi(2),2}\cdots a_{\pi(n),n}\,.$$

Using the commutativity of the product in $\mathbb{F}$ and Equation (3), we see that

$$\det A^T = \sum_{\pi\,\in\,\mathcal{S}_n} \operatorname{sign}(\pi^{-1})\ a_{1,\pi^{-1}(1)}a_{2,\pi^{-1}(2)}\cdots a_{n,\pi^{-1}(n)}\,,$$

which equals $\det(A)$ by Equation (7).

Proof of 4. Let $B = \left[A^{(\cdot,1)} \mid \cdots \mid A^{(\cdot,j)} \mid \cdots \mid A^{(\cdot,i)} \mid \cdots \mid A^{(\cdot,n)}\right]$ be the matrix obtained from A by interchanging the ith and the jth column. Then note that

$$\det(B) = \sum_{\pi\,\in\,\mathcal{S}_n} \operatorname{sign}(\pi)\ a_{1,\pi(1)}\cdots a_{j,\pi(i)}\cdots a_{i,\pi(j)}\cdots a_{n,\pi(n)}\,.$$

Define $\tilde{\pi} = \pi \circ t_{ij}$, and note that $\pi = \tilde{\pi} \circ t_{ij}$. In particular, $\pi(i) = \tilde{\pi}(j)$ and $\pi(j) = \tilde{\pi}(i)$, from which

$$\det(B) = \sum_{\pi\,\in\,\mathcal{S}_n} \operatorname{sign}(\tilde{\pi}\circ t_{ij})\ a_{1,\tilde{\pi}(1)}\cdots a_{i,\tilde{\pi}(i)}\cdots a_{j,\tilde{\pi}(j)}\cdots a_{n,\tilde{\pi}(n)}\,.$$

It follows from Equations (2) and (1) that $\operatorname{sign}(\tilde{\pi}\circ t_{ij}) = -\operatorname{sign}(\tilde{\pi})$. Thus, using Equation (6), we obtain $\det(B) = -\det(A)$.

Proof of 8. Using the standard expression for the matrix entries of the product AB in terms of the matrix entries of $A = (a_{ij})$ and $B = (b_{ij})$, we have that

$$\begin{aligned}\det(AB) &= \sum_{\pi \in \mathcal{S}_n} \operatorname{sign}(\pi) \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} a_{1,k_1} b_{k_1,\pi(1)} \cdots a_{n,k_n} b_{k_n,\pi(n)} \\ &= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} a_{1,k_1} \cdots a_{n,k_n} \sum_{\pi \in \mathcal{S}_n} \operatorname{sign}\,(\pi) b_{k_1,\pi(1)} \cdots b_{k_n,\pi(n)}.\end{aligned}$$

Note that, for fixed $k_1, \ldots, k_n \in \{1, \ldots, n\}$, the sum $\sum_{\pi \in \mathcal{S}_n} \operatorname{sign}\,(\pi) b_{k_1,\pi(1)} \cdots b_{k_n,\pi(n)}$ is the determinant of a matrix composed of rows $k_1, \ldots, k_n$ of B. Thus, by property 5, it follows that this expression vanishes unless the k_i are pairwise distinct. In other words, the sum over all choices of $k_1, \ldots, k_n$ can be restricted to those sets of indices $\sigma(1), \ldots, \sigma(n)$ that are labeled by a permutation $\sigma \in \mathcal{S}_n$. In other words,

$$\det(AB) = \sum_{\sigma \in \mathcal{S}_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \sum_{\pi \in \mathcal{S}_n} \operatorname{sign}(\pi)\; b_{\sigma(1),\pi(1)} \cdots b_{\sigma(n),\pi(n)}\,.$$

Now, proceeding with the same arguments as in the proof of Property 4 but with the role of t_{ij} replaced by an arbitrary permutation σ, we obtain

$$\det(AB) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma)\; a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \sum_{\pi \in \mathcal{S}_n} \operatorname{sign}(\pi \circ \sigma^{-1})\; b_{1,\pi\circ\sigma^{-1}(1)} \cdots b_{n,\pi\circ\sigma^{-1}(n)}\,.$$

Using Equation (6), this last expression then becomes $(\det A)(\det B)$. □

Note that Properties 3 and 4 of Theorem 6.1 effectively summarize how Elementary Row and Column Operations interact with the Determinant. These Properties together with Property 9 facilitate numerical computation of determinants for very large matrices.

7 Further Properties and Applications

There are many, many applications of Theorem 6.1. We conclude these notes with a few consequences that are particularly useful when computing with matrices. In particular, we use the determinant to characterize when a matrix can be inverted and, as a corollary, give a method for using determinants to calculate eigenvalues.

Theorem 7.1. *Let $n \in \mathbb{Z}_+$ and $A \in \mathbb{C}^{n\times n}$. Then A is invertible if and only if $\det(A) \neq 0$. Moreover, if $\det(A) \neq 0$, then $\det(A^{-1}) = (\det(A))^{-1}$.*

Proof. First assume that A is invertible so that $AA^{-1} = I$. Then, using Properties 1 and 8 of Theorem 6.1, we obtain

$$(\det(A))(\det(A^{-1})) = 1\,.$$

Therefore, $\det(A) \neq 0$ and $\det(A^{-1}) = (\det(A))^{-1}$.

It also follows from this that the determinant is invariant under basis transformation, i.e., if $T \in \mathbb{C}^{n\times n}$ is invertible, then $\det(TAT^{-1}) = \det(A)$. By Theorem 5.13 in the textbook, we know that, given any complex matrix A, there exist an invertible matrix T such that TAT^{-1} is upper triangular. Thus, by Property 9 of Theorem 6.1, $\det(A)$ is the product of the diagonal elements of this associated upper triangular matrix. Furthermore, by Theorem 5.16 in textbook, A is invertible if and only if all of the diagonal elements of the associated upper triangular matrix are non-zero. Thus, A is invertible if $\det(A) \neq 0$. □

Given a matrix $A \in \mathbb{C}^{n\times n}$ and a complex number $\lambda \in \mathbb{C}$, the expression $P(\lambda) = \det(A - \lambda I_n)$ is called the *characteristic polynomial* of A. Note that $P(\lambda)$ is a basis independent polynomial of degree n. Thus, as with the determinant, we can consider $P(\lambda)$ to be associated with the linear map that has matrix A with respect to some basis. Since the eigenvalues of A are exactly those $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible, if follows that

Corollary 7.2. *The roots of the polynomial* $P(\lambda) = \det(A - \lambda I)$ *are the eigenvalues of* A.