Notes on Solving Systems of Linear Equations

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1 From Linear Systems to Linear Maps

We begin these notes by reviewing several different conventions for denoting and studying systems of linear equations. The most fundamental of these convention involves encoding the linear system as a single matrix equation. This point of view has a long history of exploration, and numerous computational devices — including several computer programming languages — have been developed and optimized specifically for analyzing matrix equations. From the viewpoint of understanding linear systems abstractly, though, we will see that this encoding has the advantage of allowing us to further reinterpret questions about the linear system as questions about a linear map that is uniquely determined by the linear system.

1.1 From Linear Systems to Matrix Equations

Let $m, n \in \mathbb{Z}_+$ be positive integers. Then a *system* of m *linear equations* in n unknowns $x_1, \ldots, x_n$ looks like

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \tag{1}$$

where each $a_{ij}, b_i \in \mathbb{F}$ is a scalar for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In other words, each scalar $b_1, \ldots, b_m \in \mathbb{F}$ is being written as a linear combination of the unknowns $x_1, \ldots, x_n$ using coefficients from the field $\mathbb{F}$. Since the left-hand side of each equation is a linear sum, it is common to write System (1) using somewhat more compactly notation such as

$$\sum_{k=1}^{n} a_{1k}x_k = b_1, \ \sum_{k=1}^{n} a_{2k}x_k = b_2, \ \sum_{k=1}^{n} a_{3k}x_k = b_3, \ \ldots, \ \sum_{k=1}^{n} a_{mk}x_k = b_m. \tag{2}$$

To *solve* System (1) means to describe the set of all possible values for $x_1, \ldots, x_n$ (when thought of as scalars in $\mathbb{F}$) such that each of the m equations in System (1) is satisfied simultaneously. If we use $A = (a_{ij}) \in \mathbb{F}^{m\times n}$ to denote the $m \times n$ *coefficient matrix* associated to the linear system and $x = (x_i)$ to denote the $n\times 1$ column vector composed of the unknowns $x_1, \ldots, x_n$, then this is equivalent to describing the set of all vectors $x \in \mathbb{F}^n$ satisfying the single *matrix equation*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = b \qquad (3)$$

where we have used $b \in \mathbb{F}^m$ to denote the $m \times 1$ column vector formed from the right-hand sides of the equations in System (1).

It is important to note that Equation (3) differs from Systems (1) and (1) only in terms of notation. However, it is nonetheless a significantly more flexible point of view for understanding systems of linear equations. One common approach is to reinterpret solving Equation (3) as the equivalent problem of describing all coefficients $x_1, \ldots, x_n \in \mathbb{F}$ for which the following *vector equation* is satisfied:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}.$$

This approach emphasizes analysis of properties of the *column vectors* $A^{(\cdot,j)}$ $(j = 1, \ldots, n)$ of the coefficient matrix A in the matrix equation $Ax = b$. However, for the purposes of these notes, it is preferable to view solving Equation (3) as a question about the linear map $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ having so-called *canonical matrix* A, as explained in the next section.

1.2 The Canonical Matrix of a Linear Map

Let $m, n \in \mathbb{Z}_+$ be positive integers. Then, given a choice of bases for the vector spaces $\mathbb{F}^n$ and $\mathbb{F}^m$, there is a duality between matrices and linear maps. In other words, as discussed in the Notes on Linear Maps, every linear map in the set $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ uniquely corresponds to exactly one $m \times n$ matrix in $\mathbb{F}^{m\times n}$. However, you should not take this to mean that matrices and linear maps are interchangeable or indistinguishable ideas. By itself, a matrix in the set $\mathbb{F}^{m\times n}$ is nothing more than a collection of mn scalars that have been arranged in a rectangular shape. It is only when a matrix appears as part of some larger context that the theory of linear maps becomes applicable. In particular, one can gain insight into the solutions of matrix equation when the coefficient matrix is viewed as the matrix associated to a linear map under a convenient choice of bases for $\mathbb{F}^n$ and $\mathbb{F}^m$.

Given a positive integer, $k \in \mathbb{Z}_+$, one particularly convenient choice of basis for $\mathbb{F}^k$ is the so-called *standard basis* (a.k.a. the *canonical basis*) $e_1, e_2, \ldots, e_k$, where each e_i is the k-tuple having zeros for each of its component other than in the i^{th} position:

$$e_i = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \ldots, 0).$$

Then, taking the vector spaces $\mathbb{F}^n$ and $\mathbb{F}^m$ under their canonical bases, we say that the matrix $A \in \mathbb{F}^{m \times n}$ associated to the linear map $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ is the *canonical matrix* for T. One reason for this choice of basis is that it gives us the particularly nice formula

$$T(x) = Ax, \forall\, x \in \mathbb{F}^n. \tag{4}$$

In other words, one can compute the action of the linear map upon any vector in $\mathbb{F}^n$ by simply multiplying the vector by the associated canonical matrix A. There are many circumstances in which one might wish to use non-standard bases for either $\mathbb{F}^n$ or $\mathbb{F}^m$, but the trade-off is that Equation (4) will no longer hold as stated. (To modify Equation (4) for use with non-standard bases, one needs to use coordinate vectors, per the Notes on Change of Bases.)

The utility of Equation (4) cannot be overly emphasized. To get a sense of this, consider once again the generic matrix equation (Equation (3))

$$Ax = b,$$

which involves a given matrix $A = (a_{ij}) \in \mathbb{F}^{m \times n}$, a given vector $b \in \mathbb{F}^m$, and the n-tuple of unknowns x. To provide a solve to this equation means to provide a vector $x \in \mathbb{F}^n$ for which the matrix product Ax is exactly the vector b. In light of Equation (4), the question of whether such a vector $x \in \mathbb{F}^n$ exists is equivalent to asking whether or not the vector b is in the range of the linear map T.

While the encoding of System (1) into Equation (3) might be considered a matter of mere notational equivocation, the above reinterpretation of Equation (3) using linear maps is a genuine change of viewpoint. Solving System (1) (and thus Equation (3)) essentially amounts to understanding how m distinct objects interact in an ambient space having n-dimensions. (In particular, solutions to System (1) correspond to the points of intersect of m hyperplanes in $\mathbb{F}^n$.) On the other hand, questions about a linear map genuinely involve understanding a single object, i.e., the linear map itself. Such a point of view is both extremely flexible and extremely fruitful, as we will see in the sections below.

2 Solving Linear Systems

2.1 Factorizing Matrices using Gaussian Elimination

Many uses of matrices in Applied Mathematics rely on algorithms that factorize a single matrix into the product of two or more matrices, each of which has some desirable property. In this section, we discuss a particularly fundamental and computationally significant factorization for matrices that is known as *Gaussian elimination*. As a factorization algorithm, Gaussian elimination can be used to express any matrix as a product involving one matrix in so-called *reduced row-echelon form* and one or more so-called *elementary matrices*. Moreover, the underlying technique for arriving at this factorization is essentially an extension of the techniques already familiar to you for solving small systems of linear equations by hand.

Let $m, n \in \mathbb{Z}_+$ denote positive integers, and suppose that $A \in \mathbb{F}^{m\times n}$ is an $m \times n$ matrix over $\mathbb{F}$. Then, following the Notes on Matrices and Matrix Operations, recall the notation $A^{(i,\cdot)}$ and $A^{(\cdot,j)}$ for the *row vectors* and *column vectors* of A, respectively. In other other words, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$,

$$A^{(i,\cdot)} = \begin{bmatrix} a_{k1}, & \cdots, & a_{ks} \end{bmatrix} \in \mathbb{F}^{1\times s} \quad \text{and} \quad A^{(\cdot,j)} = \begin{bmatrix} a_{1\ell} \\ \vdots \\ a_{r\ell} \end{bmatrix} \in \mathbb{F}^{r\times 1}.$$

We will make extensive use of this notational decomposition of a matrix in the following series of definitions.

Definition 2.1. Let $A \in \mathbb{F}^{m\times n}$ be an $m \times n$ matrix over $\mathbb{F}$. Then we say that A is in *row-echelon form* (abbreviated *REF*) if the row vectors of A satisfy the following conditions:

(1) either $A^{(1,\cdot)}$ is the zero vector or the first non-zero entry (when read from left to right) is a one.

(2) for $i = 1, \ldots, m$, if any row vector $A^{(i,\cdot)}$ is the zero vector, then each subsequent row vector $A^{(i+1,\cdot)}, \ldots, A^{(m,\cdot)}$ is also the zero vector.

(3) for $i = 2, \ldots, m$, if some $A^{(i,\cdot)}$ is not the zero vector, then the first non-zero entry (when read from left to right) is a one and occurs to the right of the initial one in $A^{(i-1,\cdot)}$.

The initial leading one in each non-zero row is called a *pivot*. We furthermore say that A is in *reduced row-echelon form* (abbreviated *RREF*) if

(4) for each column vector $A^{(\cdot,j)}$ containing a pivot ($j = 2, \ldots, n$), the pivot is the only non-zero element in $A^{(\cdot,j)}$.

The motivation behind Definition 2.1 is that matrix equations having their coefficient matrix in RREF (and, in some sense, also REF) are particularly easy to solve. Note, in particular, that the only square matrix in RREF without zero rows is the identity matrix.

Example 2.2. The following matrices are all in REF:

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_7 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, of these, only A_4 through A_8 are in RREF as you should verify. Moreover, if we take the transpose of each of these matrices, then only A_6^T, A_7^T, and A_8^T are in RREF.

Example 2.3.

1. Consider the following matrix in RREF:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given any vector $b = \begin{bmatrix} b_1, & b_2, & b_3, & b_4 \end{bmatrix}^T \in \mathbb{F}^4$, the matrix equation $Ax = b$ corresponds to the system of equations

$$\left.\begin{array}{rcl} x_1 \qquad\qquad\qquad & = & b_1 \\ x_2 \qquad\qquad & = & b_2 \\ x_3 \qquad & = & b_3 \\ x_4 & = & b_4 \end{array}\right\}.$$

Since A is in RREF (in fact, $A = I_4$ is the 4×4 identity matrix), we can immediately conclude that the matrix equation $Ax = b$ has the solution $x = b$ for any choice of b, and, moreover, this is the only possible solution.

2. Consider the following matrix in RREF:

$$A = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given any vector $b = \begin{bmatrix} b_1, & b_2, & b_3, & b_4 \end{bmatrix}^T \in \mathbb{F}^4$, the matrix equation $Ax = b$ corresponds to the system of equations

$$\left.\begin{array}{rcrcrcrcrcrcl} x_1 & + & 6x_2 & & & & & + & 4x_5 & - & 2x_6 & = & b_1 \\ & & & & x_3 & & & + & 3x_5 & + & x_6 & = & b_2 \\ & & & & & & x_4 & + & 5x_5 & + & 2x_6 & = & b_3 \\ & & & & & & & & & & 0 & = & b_4 \end{array}\right\}.$$

Since A is in RREF, we can immediately conclude a number of facts about solutions to this system. First of all, solutions exist if and only if $b_4 = 0$. Moreover, by “solving for the pivots”, we see that the system reduces to

$$\left.\begin{array}{rclrcrcr} x_1 & = & b_1 & -6x_2 & - & 4x_5 & + & 2x_6 \\ x_3 & = & b_2 & & - & 3x_5 & - & x_6 \\ x_4 & = & b_3 & & - & 5x_5 & - & 2x_6 \end{array}\right\}.$$

In this context, x_1, x_3, and x_4 are called *leading variable* since these are the variable corresponding to the pivots in A. We similarly call x_2, x_5, and x_6 *free variables* since the leading variables have been expressed in terms of these remaining variable. Moreover, given any scalars $\alpha, \beta, \gamma \in \mathbb{F}$, it follows that the vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} b_1 - 6\alpha - 4\beta + 2\gamma \\ \alpha \\ b_2 - 3\beta \\ b_3 - 5\beta - 2\gamma \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ b_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -6\alpha \\ \alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4\beta \\ 0 \\ -3\beta \\ -5\beta \\ \beta \\ 0 \end{bmatrix} + \begin{bmatrix} 2\gamma \\ 0 \\ 0 \\ -2\gamma \\ 0 \\ \gamma \end{bmatrix}$$

must satisfy the matrix equation $Ax = b$. One can also verify that every solution to the matrix equation must be of this form. In then follows that the set of all solutions should somehow be “three dimensional”. In fact, as we will see in the sections below, the set of solutions is a so-called *affine subspace* of $\mathbb{F}^6$. In other words, it is the set of all vectors obtained by adding the vector $\begin{bmatrix} b_1, & 0, & b_2, & b_3, & 0, & 0 \end{bmatrix}^T$ to every vector in null(T), where null(T) is the subspace

$$\left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{F}^6 \mid x_1 = -6x_2 - 4x_5 + 2x_6, x_3 = -3x_5 - x_6, x_4 = -5x_5 - 2x_6\right\}$$

and T is the linear map having canonical matrix A.

As the above examples illustrate, a matrix equation having coefficient matrix in RREF corresponds to a system of equations that can be solved with only a small amount of computation. Somewhat amazingly, any matrix can be factored into a product that involves exactly one matrix in RREF and one or more of the matrices defined as follows.

Definition 2.4. A square matrix $E \in \mathbb{F}^{m\times m}$ is called an *elementary matrix* if it has one of the following forms:

1. (row exchange matrix) E is obtained from the identity matrix I_m by interchanging the row vectors $I_m^{(r,\cdot)}$ and $I_m^{(s,\cdot)}$, for some choice of positive integers $1 \le r, s \le m$. I.e., in the case that $r < s$,

$$E = \begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow r^{\text{th}} \text{ row} \\ \\ \\ \\ \leftarrow s^{\text{th}} \text{ row.} \\ \\ \\ \\ \end{array} \end{array}$$

2. (row scaling matrix) E is obtained from the identity matrix I_m by replacing the row vector $I_m^{(r,\cdot)}$ with $\alpha I_m^{(r,\cdot)}$ for some choice of scalar $\alpha \in \mathbb{F}$ and some choice of positive integer $1 \le r \le m$. I.e.,

$$E = I_m + (\alpha - 1)E_{rr} = \begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow r^{\text{th}} \text{ row} \\ \\ \\ \\ \end{array} \end{array}$$

   where E_{rr} is the standard basis vector for the vector space $F^{m\times m}$ having all entries zero except for the value one as the "r, r entry".

3. (row combination matrix) E is obtained from the identity matrix I_m by replacing the

row vector $I_m^{(r,\cdot)}$ with $I_m^{(r,\cdot)} + \alpha I_m^{(s,\cdot)}$ for some choice of scalar $\alpha \in \mathbb{F}$ and some choice of positive integers $1 \leq r, s \leq m$. I.e., in the case that $r < s$,

$$E = I_m + \alpha E_{rs} = \begin{array}{cl} \left[\begin{array}{ccccccccccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow r^{\text{th}} \text{ row} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccccccccccc} &&&&&&&&& \uparrow &&& \\ &&&&&&&&& j^{\text{th}} \text{ column} &&& \end{array} &
\end{array}$$

where E_{rs} is the standard basis vector for the vector space $F^{m \times m}$ having all entries zero except for the value one as the "r, s entry".

The "elementary" in the name "elementary matrix" comes from the correspondence between these matrices and so-called "elementary operations" on systems of equations. In particular, each of the elementary matrices are clearly invertible, just as each "elementary operation" is itself reversible. We illustrate this correspondence in the following example.

Example 2.5. Define A, x, and b by

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}.$$

We illustrate the correspondence between elementary matrices and "elementary" operations on a system of equations as follows.

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrl} 2x_1 & + & 5x_2 & + & 3x_3 = & 5 \\ x_1 & + & 2x_2 & + & 3x_3 = & 4 \\ x_1 & & & + & 8x_3 = & 9 \end{array}$	$Ax = b$

Now, to begin solving this system, one might want to either multiply the first equation through by 1/2 or interchange the first equation with one of the other equations. From a computational perspective, it is preferable to perform an interchange since multiplying through by 1/2 would unnecessarily introduce fractions. Thus, we choose to interchange the first and second equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 5 \\ x_1 & & & + & 8x_3 & = & 9 \end{array}$	$E_0Ax = E_0b, \text{ where } E_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Another reason for performing the above interchange is that it now allows us to use "row combination" operations in order to eliminate the variable x_1 from all but one of the equations. In particular, we can multiply the first equation through by -2 and add it to the second equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & - & 3x_3 & = & -3 \\ x_1 & & & + & 8x_3 & = & 9 \end{array}$	$E_1E_0Ax = E_1E_0b, \text{ where } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Similarly, in order to eliminate the variable x_1 from the third equation, we would next multiply the first equation through by -1 and add it to the third equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & - & 3x_3 & = & -3 \\ & & -2x_2 & + & 5x_3 & = & 5 \end{array}$	$E_2E_1E_0Ax = E_2E_1E_0b, \text{ where } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$

Now that the variable x_1 only appears in the first equation, we can similarly isolate the variable x_2 by multiply the second equation through by 2 and add it to the third equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & - & 3x_3 & = & -3 \\ & & & & -x_3 & = & -1 \end{array}$	$E_3 \cdots E_0Ax = E_3 \cdots E_0b, \text{ where } E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$

Finally, in order to complete the process of transforming the coefficient matrix so that it is in REF, we need only rescale row three by -1. This corresponds to multiplying the third equation through by -1 in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & - & 3x_3 & = & -3 \\ & & & & x_3 & = & 1 \end{array}$	$E_4 \cdots E_0 Ax = E_4 \cdots E_0 b, \text{ where } E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$

Now that the coefficient matrix is in REF, we can actually solve for the variables x_1, x_2, and x_3 using a process called *back substitution*. In other words, it should be clear from the third equation that

$$x_3 = 1.$$

Using this value and solving for x_2 in the second equation, it then follows that

$$x_2 = -3 + 3x_3 = -3 + 3 = 0.$$

Similarly, by solving the first equation for x_1, it follows that

$$x_1 = 4 - 2x_2 - 3x_3 = 4 - 3 = 1.$$

From a computational perspective, this process of back substitution can be applied to solve any system of equations when the coefficient matrix of the corresponding matrix equation is in REF. However, from an algorithmic perspective, it is often more useful to continue "row reducing" the coefficient matrix in order to produce a matrix in full RREF.

Here, there are several next natural step that we could perform in order to move toward RREF. Since above we worked "from the top down, from left to right", we choose to now work "from bottom up, from right to left". In other words, the first step now is to multiply the third equation through by 3 and then add it to the second equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 1 \end{array}$	$E_5 \cdots E_0 Ax = E_5 \cdots E_0 b, \text{ where } E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$

Next, we can multiply the third equation through by -3 and add it to the first equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & 1 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 1 \end{array}$	$E_6 \cdots E_0 Ax = E_6 \cdots E_0 b, \text{ where } E_6 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Finally, we can multiply the second equation through by -2 and add it to the first equation in order to obtain

System of Equations	Corresponding Matrix Equation
$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 0 \\ x_3 &= 1 \end{aligned}$	$E_7 \cdots E_0 A x = E_7 \cdots E_0 b, \text{ where } E_7 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Now it should be extremely clear that we obtained a correct solution using back substitution above. However, in many applications, it is not enough to merely find a solution. Instead, it is important either to verify that there are no other solutions or to describe the affine subspace containing all solutions as in Example 2.3(2).

To verify that this is the only solution, we appeal to the theory of linear maps. In particular, let $T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^3)$ be the linear map having canonical matrix A. Then, as discussed in Section 1.2, solving the original matrix equation $Ax = b$ corresponds to asking whether or not the vector $b \in \mathbb{F}^3$ is in the range of the T.

In order to answer this corresponding question regarding the range of T, we take a closer look at the following expression obtained from the above analysis:

$$E_7 E_6 \cdots E_1 E_0 A = I_3 \implies A = E_0^{-1} E_1^{-1} \cdots E_7^{-1} I_3,$$

where we have used the fact that each elementary matrices $E_0, \ldots, E_7$ is invertible in order to "solve" for A. In effect, this has factored A into the product of eight elementary matrices and one matrix in RREF. Moreover, from the linear map point of view, we have obtained the factorization

$$T = S_7 \circ S_6 \circ \cdots \circ S_0$$

where S_i is the (invertible) linear map having canonical matrix E_i^{-1} for $i = 0, \ldots, 7$.

This factorization of the linear map T into a composition of invertible linear maps furthermore implies that T itself is invertible. In particular, T is surjective, and so b must be an element of the range of T. Moreover, T is also injective, and so b has exactly one pre-image. Thus, the above solution to the matrix equation $Ax = b$ must also be unique.

Finally, we note that this analysis allows us to also conclude that the inverse of T has canonical matrix

$$A^{-1} = E_7 E_6 \cdots E_1 E_0 = \begin{bmatrix} 13 & -5 & -3 \\ -40 & 16 & 9 \\ 5 & -2 & 1 \end{bmatrix}.$$

Having computed this product, one could essentially "reuse" much of the above computation in order to solve the matrix equation $Ax = b'$ for several different right-hand sides $b' \in \mathbb{F}^3$. The process of "resolving" a linear system is a common practice in Applied Linear Algebra.

2.2 Solving Homogenous Linear Systems

In this section, we study solutions for an important special case of linear systems, namely *homogeneous systems*. As we will see in the next section, though, the theory of solving homogeneous systems lies at the heart of techniques for solving any linear system.

As usual, we use $m, n \in \mathbb{Z}_+$ to denote arbitrary positive integers.

Definition 2.6. The system of linear equations System (1) is called a *homogeneous system* if the right-hand side of each equation is zero. In other words, a homogeneous system corresponds to a matrix equation of the form

$$Ax = 0$$

where $A \in \mathbb{F}^{m\times n}$ is an $m \times n$ matrix and x is an n-tuple of unknowns.

In particular, based upon the discussion in Section 1.2, it should be clear that solving a homogeneous system corresponds to describing the null space $\text{null}(T)$ of the linear map $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ having canonical matrix A, where $\text{null}(T)$ is a subspace of $\mathbb{F}^n$. In describing $\text{null}(T)$, there are three important cases to keep in mind:

Definition 2.7. The system of linear equations System (1) is called

1. *overdetermined* if $m > n$.
2. *square* if $m = n$.
3. *underdetermined* if $m < n$.

For your own practice, you should provide a proof of the following theorem using the theory of linear maps.

Theorem 2.8. *Every homogeneous system of linear equations has a solution, namely the zero vector. Moreover, every underdetermined homogenous has infinitely many solution.*

We call the zero vector the *trivial solution* of a homogeneous linear system. The fact that every homogeneous linear system has the trivial solution is equivalent to the fact that the image of the zero vector under any linear map always results in the zero vector. Thus, solving a homogenous linear system reduces to determining if solutions other than the trivial solution exist. Furthermore, determining whether or not the trivial solution is unique then becomes a dimensionality question from the linear map point of view.

One method for studying the null space of a linear map is to first use Gaussian elimination (as demonstrated in Example 2.5) in order to factor the canonical matrix of the linear map. Then, because the corresponding system of equations is homogeneous, the RREF matrix obtained will have the same solutions as the original matrix equation. Thus, it suffices to study the null space of the linear map corresponding to this RREF matrix. This simplification is valid because each elementary matrices obtained when factoring a matrix via Gaussian elimination is invertible. In other words, if a given matrix A satisfies

$$E_k E_{k-1} \cdots E_0 A = A_0,$$

where each E_i an elementary matrix and A_0 is an RREF matrix, then the matrix equation $Ax = 0$ has the exact same solution set as $A_0 x = 0$ since

$$E_0^{-1} E_1^{-1} \cdots E_7^{-1} 0 = 0.$$

Put another way, if T is the linear map having canonical matrix A and T_0 is the linear map having canonical matrix A_0, then $\mathrm{null}(T) = \mathrm{null}(T_0)$.

In the following examples, we illustrate the process of determining the null space for a linear map having associated matrix in RREF.

Example 2.9.

1. Consider the matrix equation $Ax = 0$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   This corresponds to an overdetermined homogeneous system of linear equations. Moreover, since there are no free variables (as defined in Example 2.3), it should be clear that this system has only the trivial solution.

2. Consider the matrix equation $Ax = 0$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   This corresponds to an overdetermined homogeneous system of linear equations. Unlike the above example, we see that x_3 is a free variable for this system. Thus, denoting by T the linear map having canonical matrix A, one can solve for the leading variables x_1 and x_2 (as illustrated in Example 2.3) in order to obtain $\mathrm{null}(T)$ as the one-dimensional subspace of $\mathbb{F}^3$ given by

$$\mathrm{null}(T) = \left\{ (x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 = -x_3, x_2 = -x_3 \right\}.$$

3. Consider the matrix equation $Ax = 0$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   This corresponds to a square homogeneous system of linear equations with two free variables. Thus, as above, we denote by T the linear map having canonical matrix A and solve for the leading variable x_1 in order to obtain null(T) as the two-dimensional subspace of $\mathbb{F}^3$ given by

$$\mathrm{null}(T) = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + x_2 + x_3 = 0\right\}.$$

2.3 Solving Inhomogeneous Linear Systems

In this section, we conclude our study of solutions for linear systems by relating the analysis of the general case to the analysis of homogeneous system as discussed in the previous section. In particular, we will see that it takes little more work to solve a general linear system than it does to solve a homogeneous one.

As usual, we use $m, n \in \mathbb{Z}_+$ to denote arbitrary positive integers.

Definition 2.10. The system of linear equations System (1) is called an *inhomogeneous system* if the right-hand side of at least one equation is not zero. In other words, a homogeneous system corresponds to a matrix equation of the form

$$Ax = b$$

where $A \in \mathbb{F}^{m\times n}$ is an $m \times n$ matrix, x is an n-tuple of unknowns, and $b \in \mathbb{F}^m$ is a vector having at least one non-zero component.

As illustrated in Example 2.3 above, the set of solutions to an inhomogeneous system of linear equations does not form a subspace of $\mathbb{F}^n$. Instead, given any non-zero vector $b \in \mathbb{F}^m$, the set of all solutions to $Ax = b$ consists of the set of all pull-backs $u \in \mathbb{F}^n$ for b under T, where T is the linear map having canonical matrix A. Thus, as mentioned in Section 1.2 above, the matrix equation $Ax = b$ has a solution for every vector $b \in \mathbb{F}^m$ exactly when T is surjective. Consequently, an overdetermined inhomogeneous system will necessarily be unsolvable for certain choices of right-hand side. On the other hand, if T is an invertible linear map (as in Example 2.5), then this solution is always unique since the set $\{u \in \mathbb{F}^n \mid T(v) = b\}$ will have exactly one element for every choice of $b \in \mathbb{F}^m$.

It turns out that, given any linear map $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ and any non-zero vector $b \in \mathbb{F}^m$, the structure of the set $\{v \in \mathbb{F}^n \mid T(v) = b\}$ is highly dependent upon null(T). We illustrate this in the following theorem, for which you should provide a proof as practice using the techniques discussed in these notes.

Theorem 2.11. *Let $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ be a linear map, let $b \in \mathbb{F}^m$ be a non-zero vector, and denote $U = \{u \in \mathbb{F}^n \mid T(v) = b\}$. Then given any element $u \in U$, we have that*

$$U = null(T) + u = \{v + u \mid v \in null(T)\}$$

In particular, the set U constructed above is called an *affine subspace* since it is a genuine subspace of $\mathbb{F}^n$ that has been "offset" by the vector $u \in \mathbb{F}^n$. Any set having this structure might also be called a *coset* (when used in the context of Group Theory) or a *linear manifold* (when used in a geometric context such as a discussion of intersection hyperplanes).

This decomposition of the set of solutions to an inhomogeneous system of linear equations allows us to exploit the simplicity of describing solutions to homogeneous systems as detailed in the section above. Given an $m \times n$ matrix $A \in \mathbb{F}^{m\times n}$ and a non-zero vector $b \in \mathbb{F}^m$, we call $Ax = 0$ the *associated homogeneous system* to the inhomogeneous system $Ax = b$. Then, to solve $Ax = b$, one first uses Gaussian elimination to factorize A. As discussed in the previous section, this factorization allows us to immediately describe the set of solutions to $Ax = 0$. Given this solution set, it then suffices to find any so-called *particular solution* to $Ax = b$ in order to describe all possible to $Ax = b$. This then results in an affine subspace of solutions as illustrated in Example 2.3.

We further illustrate this process in the following examples.

Example 2.12.

1. Consider the matrix equation $Ax = b$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   and $b \in \mathbb{F}^4$. This corresponds to an overdetermined inhomogeneous system of linear equations. Note, in particular, that the bottom row $A^{(4,\cdot)}$ of the matrix corresponds to the equation $0 = b_4$, from which $Ax = b$ has no solution unless the fourth component of the vector b is zero. Furthermore, we conclude from the remaining rows of the matrix that $x_1 = b_1$, $x_2 = b_2$, and $x_3 = b_3$. Thus, given any vector b having fourth component zero, we conclude that this solution is unique.

2. Consider the matrix equation $Ax = b$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $b \in \mathbb{F}^4$. This corresponds to an overdetermined inhomogeneous system of linear equations. Note, in particular, that the bottom two rows of the matrix corresponds to the equations $0 = b_3$ and $0 = b_4$, from which $Ax = b$ has no solution unless the third and fourth component of the vector b are both zero. Furthermore, we conclude from the remaining rows of the matrix that $x_1 = b_1 - x_3$ and $x_2 = b_2 - x_3$. In particular, x_3 is a free variable for this system. Thus, denoting by T the linear map having canonical matrix A, one can solve for the leading variables x_1 and x_2 in order to first obtain $\text{null}(T)$ as the one-dimensional subspace of $\mathbb{F}^3$ given by

$$\text{null}(T) = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 = -x_3, x_2 = -x_3\right\}.$$

Then, since $x = \begin{bmatrix} b_1, & b_2, & 0 \end{bmatrix}^T \in \mathbb{F}^3$ is a particular solution for $Ax = b$ (which was obtained by arbitrarily setting the free variable $x_3 = 0$), it follows that the set of all solutions to the matrix equation $Ax = b$ has the form

$$\left\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 = b_1 - x_3, x_2 = b_2 - x_3\right\}.$$

3. Consider the matrix equation $Ax = b$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $b \in \mathbb{F}^4$. This corresponds to a square inhomogeneous system of linear equations with two free variables. Thus, as above, this system has no solutions unless $b_2 = b_3 = 0$. Moreover, denoting by T the linear map having canonical matrix A and solving for the leading variable x_1, we obtain $\text{null}(T)$ as the two-dimensional subspace of $\mathbb{F}^3$ given by

$$\text{null}(T) = \left\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + x_2 + x_3 = 0\right\}.$$

Then, since $x = \begin{bmatrix} b_1, & 0, & 0 \end{bmatrix}^T \in \mathbb{F}^3$ is clearly a particular solution for $Ax = b$, it follows that the set of all solutions to the matrix equation $Ax = b$ has the form

$$\left\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + x_2 + x_3 = b_1\right\}.$$