# The Language of Sets and Functions

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(January 7, 2007)

## 1 The Language of Sets

### 1.1 Definition and Notation

We begin by defining a *set* to be any *well-defined* collection of *distinct* objects. This means two things: there is no ambiguity in deciding whether or not a given object belongs to a set, and the objects in a set must be distinguishable from each other.

**Definition 1.1.** A *set* $S$ is any (unordered) collection of (distinct) objects $s$ whose membership in $S$ is well-defined. Given an object $s$, we say that *$s$ is an element of $S$*, denoted $s \in S$, if $s$ is any object inside $S$. Otherwise, we write $s \notin S$.

**Example 1.2.** Some examples of sets include

1. the *empty set* (a.k.a the *null set*), which is denote by $\{\}$ or $\emptyset$. This is the set with no objects inside of it, which is certainly valid under the definition of set. In particular, given any object $s$, $s \notin \emptyset$.

2. so-called *singleton sets*. These are sets that contain only a single element.

3. the set $\{a, b, c\}$ containing the first three lower case English letters. Since the elements in a set are unordered, we could also write this set as $\{a, c, b\}$ or $\{c, b, a\}$, etc.

   Note that the elements in a set are also required to be distinct. Given this requirement, something like $\{a, b, a, c\}$ would not be considered a set. However, it is often convenient to just agree that $\{a, b, a, c\} = \{a, b, c\}$ unless the context dictates otherwise.

4. the sets $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, of integers, real numbers, and complex numbers, respectfully.

5. the single set $\{\mathbb{Z}, \mathbb{R}, \mathbb{C}\}$ that contains as object the sets of numbers from the last example. There is nothing in the definition that restricts objects from being also sets.

6. the set $\mathbb{B}$ of all US bookstores holding a book sale at a fixed moment in time.

   Even though it would most likely be quite difficult to explicitly list its elements, $\mathbb{B}$ nonetheless qualifies as a set. In particular, one could determine whether or not a particular bookstore $b$ is an element of $\mathbb{B}$ by telephoning them to ask about book sales.

   At the same time, though, note that the collection $\mathbb{B}'$ of all *interesting* US bookstores holding sales during a fixed moment in time would not be a set. The problem with $\mathbb{B}'$ is that there is no well-defined membership rule unless we can first rigorously define what it means for a bookstore to be "interesting".

Note that these two bookstore examples only make sense if we first define the set of all US bookstores $\mathbb{U}$ so that any particular bookstore $b \in \mathbb{U}$ is well-defined as an object before being tested for membership in a set like $\mathbb{B}$. More generally, there is an all-encompassing *universal set*, often dictated implicitly by the context, that contains every object $s$ that we might wish to test for membership in a set $S$. A common practice is then to specify $S$ by giving some type of pattern or constructive algorithm that distinguishing objects within the universal set. We illustrate three most common methods for this in the following example.

**Example 1.3.**

1. The simplest form of pattern uses *list notation* in order to either explicitly or implicitly list every element in a set. The set $\{a, b, c\}$ from Example **??**(1) above is an example of the former, and the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ from Example **??**(2) is an example of the latter.

2. A more involved method of specifying a set uses something called *set builder notation*. Here, a generic object from some universal set is restricted by an explicit condition in order to specify elements. An example is the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

   where we start by taking the entire universe of (admittedly ill-defined) fractions $a/b$ and then restrict to the case that each fraction is formed over the integers without zero division. This notation should be read something like

$$\mathbb{Q} \text{ is the set of all fractions } \frac{a}{b} \text{ such that } a \text{ and } b \text{ are integers and } b \neq 0.$$

3. Finally, when dealing with numbers, a common variant of set builder notation is *interval notation*. Here, elements are selected from a universal set based upon their relative size or rank. For example,

$$(1, 3] = \{x \in \mathbb{R} \mid 1 < x \leq 3\}.$$

## 1.2 Operations and Relations on Sets

Given how fundamental sets are to the mathematical thought process, you should not be surprised to know that there is a rich vocabulary for describing how sets can be related to each other and operated upon by other sets. In this section, we briefly look at the most basic of such concepts.

**Definition 1.4.** Given two sets $S$ and $T$, we say that $S$ is a *subset* of $T$ or that $S$ is *contained in* $T$ (denoted $S \subset T$ or $T \supset S$) if, given any element $s \in S$, $s$ is also an element of $T$. Moreover, if $S \subset T$ but $S \neq T$, then we call $S$ a *proper* subset of $T$.

We summarize some of the basic properties of set containment as follows.

**Theorem 1.5.** *Let $R$, $S$, and $T$ be sets. Then*

1. *$\emptyset \subset R$.*
2. *(reflexivity) $R \subset R$.*
3. *(antisymmetry) If $R \subset S$ and $S \subset R$, then $R = S$.*
4. *(transitivity) If $R \subset S$ and $S \subset T$, then $R \subset T$.*

We also define the following operations on sets:

**Definition 1.6.** Let $S, T$ be sets with respect to some universal set $U$. Then we define

1. the *union* of $S$ and $T$ to be the set $S \cup T = \{x \in U \mid x \in S \text{ or } x \in T\}$.
2. the *intersection* of $S$ and $T$ to be the set $S \cap T = \{x \in U \mid x \in S \text{ and } x \in T\}$.
3. the *complement* of $S$ to be the set $\widetilde{S} = \{x \in U \mid x \notin S\}$.
4. the *set difference* of $S$ from $T$ to be the set $T \setminus S = \{x \in U \mid x \in T \text{ but } x \notin S\}$.

There are many, many ways in which these operations interact. We summarize some of the most essential properties as follows:

**Theorem 1.7.** *Let $R, S, T$ be sets. Then*

1. *(distributivity) $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ and $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$.*
2. *(De Morgan's Laws) $\widetilde{A \cup B} = \widetilde{A} \cap \widetilde{B}$ and $\widetilde{A \cap B} = \widetilde{A} \cup \widetilde{B}$.*
3. *(relative complements) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

We close this section with another fundamental operation on sets. Unlike the above operations on sets, though, the *Cartesian product* of two sets builds a set of entirely new objects and is not based upon comparison. In particular, given sets $S$ and $T$, their Cartesian product is the set $S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}$ of all ordered pairs that can be formed with elements from $S$ in the first position and elements from $T$ in the second position.

# 2 The Language of Functions

## 2.1 Definition and Notation

While sets are already a useful concept, their utter ubiquity and indispensability in mathematical problem solving comes from the definition of a function. In particular, a function is a special type of relationship between two sets that we define as follows.

**Definition 2.1.** A *function* $f : X \to Y$ is a rule that assigns to each element $x$ in a set $X$ called the *domain* exactly one element $y$ in a set $Y$ called the *codomain*. Moreover, if $x \in X$ is assigned the value $y \in Y$, then we call $y$ the *image* of $x$ under $f$ and denote this relationship by $x = f(y)$.

We can also associate to every function its *graph*, meaning the set $\{(x, f(x)) \mid x \in X\}$, which is a subset of the Cartesian product $X \times Y$. In particular, we can define the *range* of a function $f$ to be the set $\mathrm{Ran}(f) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$. Then the graph of $f$ is exactly $X \times \mathrm{Ran}(f) \subset X \times Y$. We also define the *pre-image* (a.k.a. *pullback*) of each individual $y \in Y$ as the set $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

## 2.2 Some Special Types of Functions

There are three basic types of functions, each of which distinctly places extra conditions on how elements in the domain are assigned values in the codomain. The resulting fundamental classes of functions are summarized in the following definition.

**Definition 2.2.** Let $f : X \to Y$ be a function. Then we call $f$

1. an *injection* (a.k.a. a *one-to-one function*) if $f$ assigns at most one element from the domain to each element in the codomain. In other words, $f$ is an injection if, for each $y \in Y$, the cardinality $|f^{-1}(y)|$ of the pullback of $y$ is at most one.

2. a *surjection* (a.k.a. an *onto function*) if $f$ assigns at least one element from the domain to each element in the codomain. In other words, $f$ is a surjection if, for each $y \in Y$, the cardinality $|f^{-1}(y)|$ of the pullback of $y$ is at least one.

3. a *bijection* (a.k.a. an *invertible function*) if $f$ is both an injection and a surjection. In other words, $f$ is a bijection if, for each $y \in Y$, the cardinality $|f^{-1}(y)|$ of the pullback of $y$ is exactly one.

In particular, if $f$ is a bijection, then the assignment of values from the domain to the codomain is called a *one-to-one correspondence* since exactly one element in the domain corresponds to exactly one element in the codomain. Consequently, it is then possible to literally "undo" the assignment of values under $f$. This yields the *inverse* of the function $f$, which we denote by $f^{-1}$. Since each pullback of a bijection is a singleton set, this is only a minor abuse of notation.

We conclude this section with some distinguished examples of functions that arise frequently in mathematics.

**Example 2.3.** Let $X$ and $Y$ be sets. Then,

1. for each $y \in Y$, we can define the *constant map* $\mathrm{const}_y : X \to Y$ whose values are given by $\mathrm{const}_y(x) = y$ for each $x \in X$. Note in particular that $\mathrm{const}_y$ is one-to-one if and only if $Y = \{y\}$ is a singleton set, and $\mathrm{const}_y$ is onto if and only if $X = \{x\}$ is a singleton set. It follows that $\mathrm{const}_y$ is invertible if and only if $X = Y = \{y\}$. But then $\mathrm{const}_y$ is just a special case of the identity map, which we define next.

2. if $X = Y$, we can define the *identity* function $\mathrm{id}_X : X \to Y$ whose values are given by $\mathrm{id}_X(x) = x$ for each $x \in X$. Clearly $\mathrm{id}_X$ is a bijection regardless of the structure on $X$, and, moreover, $\mathrm{id}_X$ is equal to its own inverse function $\mathrm{id}_X^{-1}$.

3. if $X \subset Y$, we can define the *inclusion map* $\mathrm{incl}_X : X \to Y$ whose values are given by $\mathrm{incl}_X(x) = x$ for each $x \in X$. Clearly $\mathrm{incl}_X$ is a one-to-one function, but it cannot be onto unless $X = Y$.

## 2.3 Operations on Functions

We conclude these notes by defining the following two operations on functions. Both will see many applications throughout your study of Linear Algebra.

**Definition 2.4.** Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

1. we can define the *composition* of $f$ with $g$ as the function $g \circ f : X \to Z$ whose values are given by $(g \circ f)(x) = g(f(x))$ for each $x \in X$.

2. we can define the *restriction* of $f$ to a subset $W \subset X$ as the function $f|_W : X \to Y$ whose values are given by $f|_W(w) = f(w)$ for each $w \in W$.

In particular, composition interacts with invertible functions in way that might appear somewhat unnatural at first.

**Theorem 2.5.** *Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Then*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

To remember this result, you should sit down and draw a diagram in order to more clearly see how inverting functions as stated reverses the order in which the sets $X$, $Y$, and $Z$ are visited by the resulting functions.