LECTURE 12: DIVIDED DIFFERENCE OPERATORS

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Definition 0.1. *Let $X = \{x_1, \cdots, x_n\}$, then $s_i \in S_n$ acts on $f \in \mathbb{Z}[X]$ by switching the x_i and x_{i+1}. That is,*

$$s_i.\, f(x_1, \cdots, x_i, x_{i+1}, \cdots, x_n) = f(x_1, \cdots, x_{i+1}, x_i, \cdots, x_n)$$

For $s_i \in S_n$, define $\partial_i : \mathbb{Z}[X] \to \mathbb{Z}[X]$ by

$$(\partial_i f)(x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n) - s_i.\, f(x_1, \cdots, x_n)}{x_i - x_{i+1}}$$

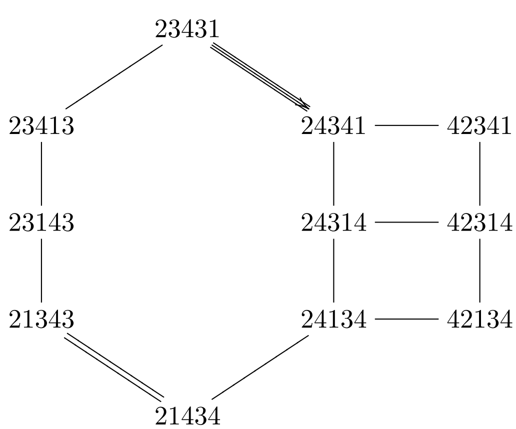
In another word, $\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$.

1. Graph of reduced words

Given $w \in (W, S)$, we can define a colored graph $\Gamma(w)$, called the **graph of reduced words** of w, as follow. The nodes in this graph are the set of all reduced words of w. Let u, v be two nodes of this graph, (i.e, two reduced work of w) then u, v are connected by an edge colored (labeled) by a defining relation of (W, S) if and only if u can be transformed to v (or vise versa) by one application of the given defining relation.

It is clear that the defining relations $s_i^2 = e$ are never used in $\Gamma(w)$, for all nodes in $\Gamma(w)$ are reduced words.

Example 1.1. *Let the Coxeter system be $(\mathcal{S}_5), \{s_1, s_2, s_3, s_4\})$, and let $w = [31542]$ in one-line notation, then one reduced word for w could be $s_2 s_3 s_4 s_3 s_1$, we will just code this word by its indices 23431. Let us show the connected component of $\Gamma(w)$ that contains this word. (As we will see later this is the whole graph)*



Proposition 1.2. *Given $w \in (W, S)$, $\Gamma(w)$ is connected.*

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Proof. We prove the statement for the case of $(\mathcal{S}_n, \{s_1, \cdots, s_{n-1}\})$, the spirits are the same for all other types of Coxeter group.

Induct on $\ell(w)$. For the case $\ell(w) \leq 2$ the statement clearly holds.

Now assume $\ell(w) \geq 3$, let $\underline{a} = a_1 \cdots a_l$ and $\underline{b} = b_1 \cdots b_l$ be two reduced words of w. By E.P. $b_1 w$ has a reduced expression of the form $a_1 \cdots \hat{a_k} \cdots a_l$, for some $1 \leq k \leq l$. Thus $b_1 a_1 \cdots \hat{a_k} \cdots a_l$ is another reduced expression of w.

First note that by induction, $b_2 \cdots b_l$ and $a_1 \cdots \hat{a_k} \cdots a_l$ are connected by a sequence of edges in $\Gamma(b_1 w)$, thus $b_1 b_2 \cdots b_l$ and $b_1 \cdots \hat{a_k} \cdots a_l$ are connected by a sequence of edges in $\Gamma(w)$.

If $k < l$, then in $\Gamma(w a_l)$, $a_1 \cdots a_{l-1}$ and $b_1 \cdots \hat{a_k} \cdots a_{l-1}$ are connected by a sequence of edges, thus in $\Gamma(w)$, $a_1 \cdots a_l$ and $b_1 \cdots \hat{a_k} \cdots a_l$ are connected by a sequence of edges. Therefore $\underline{a}$ and $\underline{b}$ are connected in $\Gamma(w)$.

If $k = l$, then either a_1 and b_1 are consecutive or not. If they are not, then $b_1 a_1 \cdots a_{l-1}$ and $a_1 b_1 a_2 \cdots a_{l-1}$ are connected by a single edge labeled by $a_1 b_1 = b_1 a_1$. Now in $\Gamma(w)$, $\underline{a}$ and $a_1 b_1 a_2 \cdots a_{l-1}$ are connected by a sequence of edges "lifted" from $\Gamma(a_1 w)$, thus $\underline{a}$ and $\underline{b}$ are connected in $\Gamma(w)$.

Finally, if $k = l$ but a_1 and b_1 are consecutive, then by E.P. $a_1 b_1 a_1 \cdots \hat{a_j} \cdots a_{l-1}$ for some $1 \leq j \leq l-1$ is another reduced word of w. If $j = 1, \cdots, l-2$ then we just repeat about argument for the case $k < l$ with $\underline{b'} = \underline{a}$ and $\underline{a'} = b_1 a_1 \cdots a_{l-1}$. Otherwise if $j = l-1$, in particular $j > 1$, then $a_1 b_1 a_1 \cdots \hat{a_j} \cdots a_{l-1}$ is connected to $b_1 a_1 b_1 \cdots \hat{a_j} \cdots a_{l-1}$ by an edge labeled by $a_1 b_1 a_1 = b_1 a_1 b_1$ in $\Gamma(w)$. Now we note that $\underline{b}$ and $b_1 a_1 b_1 \cdots \hat{a_j} \cdots a_{l-1}$ are connected in $\Gamma(w)$ by lifting a path from $\Gamma(b_1 w)$, and $\underline{a}$ and $a_1 b_1 a_1 \cdots \hat{a_j} \cdots a_{l-1}$ are connected in $\Gamma(w)$ by lifting a path from $\Gamma(a_1 w)$, we are done. □

2. Properties of divided difference operators

Lemma 2.1. *If $f, g \in \mathbb{Z}[X]$ then*

$$\partial_i (f * g) = (\partial_i f) * g + (s_i .\, f) * (\partial_i g)$$

Proof.

$$\begin{aligned}
\partial_i (f * g) &= \frac{f * g - s_i .\, (f * g)}{x_i - x_{i+1}} \\
&= \frac{f * g - (s_i .\, f) * g + (s_i .\, f) * g - s_i .\, (f * g)}{x_i - x_{i+1}} \\
&= \frac{f * g - (s_i .\, f) * g}{x_i - x_{i+1}} + \frac{(s_i .\, f) * g - (s_i .\, f) * (s_i .\, g)}{x_i - x_{i+1}} \\
&= (\partial_i f) * g + (s_i .\, f) * (\partial_i g)
\end{aligned}$$

□

Theorem 2.2 (Nil-Coxeter relations)**.**

$$\partial_i \partial_j = \partial_j \partial_i \text{ for } |i - j| > 1$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \text{ for } i = 1, \cdots, n-1$$
$$\partial_i^2 = 0$$

Proof. In lecture note 1, Steven and Alex gave a detailed proof. □

Definition 2.3. *If $a_1 \cdots a_l$ is a reduced word of $w \in \mathcal{S}_n$, then define $\partial_w = \partial_{a_1} \cdots \partial_{a_l}$.*

Remark 2.4. *Because of the fact that $\Gamma(w)$ is connected (where edges in Γ correspond to only the first two nil-Coxeter relations), ∂_w does not depends on the choice of reduced word, thus is well-defined.*

On the other hand, if $a_1 \cdots a_l$ is not a reduced word, then $\partial_{a_1} \cdots \partial_{a_l} = 0$. To see that we let $1 \leq j < l$ be such that $u = a_1 \cdots a_j$ is a reduced word but $a_1 \cdots a_{j+1}$ is no longer reduced. Then there is another reduced expression of u that is ended with a_{j+1}: $b_1 \cdots a_{j+1}$. Now $\partial_{a_1} \cdots \partial_{a_j} \partial_{a_{j+1}} \partial_{a_l} = \partial_{b_1} \cdots \partial_{a_{j+1}} \partial_{a_{j+1}} \cdots \partial_{a_l} = 0$ by the third relation.

From above consideration, we can conclude

$$(*) \qquad \partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.5. *$s_i \circ \partial_w = \partial_w$ if and only if $\ell(iw) = \ell(w) - 1$*

Proof.

$$\begin{aligned} s_i \circ \partial_w &= \partial_w \Leftrightarrow \partial_i \partial_w = 0 \text{ (by definition)} \\ \Leftrightarrow \ell(s_i w) = \ell(w) - 1 \text{ (by *)} \end{aligned}$$

□

Proposition 2.6. *If w_0 is the longest element of $\mathcal{S}_n$, then*

$$\partial_{w_0} = a_\delta^{-1} \sum_{w \in \mathcal{S}_n} \epsilon(w) w$$

where

$$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

is the Vandermond determinant, and $\epsilon(w)$ is the sign of w.

Proof. By definition, for any $v \in \mathcal{S}_n$, ∂_v can be written of the form

$$\partial_v = \sum_{w \in \mathcal{S}_n} c_{v,w} w$$

In particular, we can write $\partial_w = \sum_{w \in \mathcal{S}_n} c_w w$. By above lemma, we have $s_i \partial_{w_0} = \partial_{w_0}$ for any $i = 1, \cdots, n-1$, thus $u \partial_{w_0} = \partial_{w_0}$ for any $u \in \mathcal{S}_n$. This implies

$$\sum_{w \in \mathcal{S}_n} c_w w = u \sum_{w \in \mathcal{S}_n} c_w w = \sum_{w \in \mathcal{S}_n} (u.\, c_w)(uw)$$

Comparing the coefficient, we get $c_{uw} = u.\, c_w$. Thus if we know the coefficient c_w for some w, then we can derive c_w for all w. Indeed, we claim that we know

$$c_{w_0} = a_\delta^{-1} \epsilon(w_0)$$

Assume above claim, we note $w = w w_0 w_0$, thus $c_w = w w_0 c_{w_0} = \epsilon(w w_0) \epsilon(w_0) a_\delta^{-1} = \epsilon(w) a_\delta^{-1}$.

The only thing left to show is the claim. One reduced expression of w_0 is

$$w_0 = (s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1})$$

So,

$$\partial_{w_0} = (\partial_{n-1} \cdots \partial_1)(\partial_{n-1} \cdots \partial_2) \cdots (\partial_{n-1})$$

We are interested in the coefficient of w_0 after "multiply out" the rhs. For $n = 3$, the least non-trivial case we see

$$\partial_{w_0} = \partial_2\partial_1\partial_2 = \frac{1}{x_2 - x_3}(1 - s_2)\frac{1}{x_1 - x_2}(1 - s_1)\frac{1}{x_2 - x_3}(1 - s_2)$$

clearly $c_{w_0} = \frac{1}{x_2-x_3}\frac{1}{x_1-x_2}\frac{1}{x_2-x_3}(-1)^3$, claim shown.

The general case can be checked explicitly in a similar fashion. $\square$