LECTURE 22: STANDARD K-TABLEAUX AND GRASSMANNIAN AFFINE PERMUTATIONS

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The main result of this note is the bijection between {standard k-tableaux} $\leftrightarrow$ {reduced words of Grassmannian affine permutations}.

We will define these two sets and then describe the bijection.

Remark 0.1. *Throughout this lecture note, unless otherwise stated, $n = k + 1$.*

1. Standard k-tableaux

Definition 1.1 (Standard k-tableaux)**.** *Let γ be a $(k+1)$-core, $\lambda = \mathrm{b}(\gamma)$. Let $m = |\lambda|$. A **standard k-tableau** of shape γ is a filling of γ with $\{1, \dots, m\}$ such that the following two conditions are satisfied:*

1. *the filling is row strict and column strict;*
2. *repeated letters have the same $(k+1)$-residue.*

Example 1.2. *Let $k = 2$, $\gamma = (4, 2, 2, 1, 1)$ is a 3-core, $\lambda = \mathrm{b}(\gamma) = (2, 1, 1, 1, 1)$ with $m = |\lambda| = 6$. Then the following tableau is a standard 2-tableau:*

1	2	3	5
3	5		
4	6		
5			
6			

It is clear that each row/column is strictly increasing. The only letter '1' is at $(1, 1)$ position which has 3-residue 0. The only letter '2' is at $(1, 2)$ position which has 3-residue 1. There are two letter '3's, at position $(1, 3)$ and $(2, 1)$, both of which have 3-residue 2. One letter '4' of residue 1. Two letter '5's of residue 0. Finally, two letter '6' of residue 2.

Denote the set of all n-cores by $\mathcal{C}_n$.

2. Grassmannian affine permutations via an action of $\widetilde{S_n}$ on $\mathcal{C}_n$

In this section we discuss a correspondence of Grassmannian affine permutations with cores using an action of $\widetilde{S_n}$ on $\mathcal{C}_n$. Before we can do this, we need to say a bit more about $\mathcal{C}_n$.

Recall that given k, each box (i, j) is assigned a k-residue given by $j - i \pmod{k+1}$.

Definition 2.1. *Given a partition p, define $A_k(p, \ell)$ to be the set of all boxes of k-residue ℓ that are not already in p, but when added to p the result remains a partition. The elements in $A_k(p, \ell)$ are called the addable corners of (k-)residue ℓ. Similarly, define $R_k(p, \ell)$ to be the set of all boxes of k-residue ℓ that are already in p and when removed from p the result remains a partition. The elements in $R_k(p, \ell)$ are called the removable corners of (k-)residue ℓ.*

Theorem 2.2. *For $p \in \mathcal{C}_n$, either $A_k(p,\ell) = \emptyset$ or $R_k(p,\ell) = \emptyset$ for any ℓ. Moreover, $p \cup A_k(p,\ell) \in \mathcal{C}_n$ and $p \setminus R_k(p,\ell) \in \mathcal{C}_n$.*

Proof. This can be checked explicitly. $\square$

Definition 2.3 (Simple reflections of $\widetilde{S_n}$ acting on $\mathcal{C}_n$)**.** *Let $s_i \in \widetilde{S_n}$ be a simple reflection, let $\gamma \in \mathcal{C}_n$, then define*

$$s_i.\gamma = \begin{cases} \gamma \cup A_k(\gamma,i) & A_k(\gamma,i) \neq \emptyset \\ \gamma \setminus R_k(\gamma,i) & R_k(\gamma,i) \neq \emptyset \\ \gamma & \textit{otherwise} \end{cases}$$

Example 2.4. *Let $k = 2$ (thus $n = 3$), then*

$$s_1 . \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & & \\ \cline{1-2} 1 & 2 & & \\ \cline{1-2} 0 & & & \\ \cline{1-1} 2 & & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & & \\ \cline{1-3} 1 & 2 & & & \\ \cline{1-2} 0 & 1 & & & \\ \cline{1-2} 2 & & & & \\ \cline{1-1} 1 & & & & \\ \cline{1-1} \end{array}$$

$$s_2 . \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & & \\ \cline{1-2} 1 & 2 & & \\ \cline{1-2} 0 & & & \\ \cline{1-1} 2 & & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & & \\ \cline{1-2} 1 & & & \\ \cline{1-1} 0 & & & \\ \cline{1-1} \end{array}$$

$$s_0 . \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & & \\ \cline{1-2} 1 & 2 & & \\ \cline{1-2} 0 & & & \\ \cline{1-1} 2 & & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & \\ \cline{1-2} 1 & 2 & \\ \cline{1-2} 0 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} \end{array}$$

Proposition 2.5. *Definition 2.3 defines an action of $\widetilde{S_n}$ on $\mathcal{C}_n$.*

Proof. $\{s_i \mid i = 0, 1, \ldots, n-1\}$ acting on an n-core γ satisfy the braid relation, that is

(1) $s_i^2 \cdot \gamma = \gamma$ for $i = 0, 1, \ldots, n-1$;
(2) $s_i s_j \cdot \gamma = s_j s_i \cdot \gamma$ for $|i-j| > 1$;
(3) $s_i s_{i+1} s_i \cdot \gamma = s_{i+1} s_i s_{i+1} \cdot \gamma$ for $i = 0, 1, \ldots, n-1$, where addition on indices is defined in $\mathbb{Z}_n$.

All above can be easily verified by using the abacus representation of $\mathcal{C}_n$. The original proof of above result is in the paper "Ordering the affine symmetric group" by Lascoux (http://phalanstere.univ-mlv.fr/ al/pub_engl.html). $\square$

The action defined above is transitive but not simple, it is then natural to consider the stabilizer of the "special" element $\emptyset \in \mathcal{C}_n$. It is easily seen that $STAB_{\widetilde{S_n}}(\emptyset) = S_n$. Thus the map $\mathcal{C} : \widetilde{S_n}/S_n \to \mathcal{C}_n$ induced by above action is a bijection. Indeed, $\widetilde{S_n}/S_n$ is the set of **Grassmannian affine permutations**.

There is another point of view of $\widetilde{S_n}/S_n$. If we treat S_n as a parabolic subgroup of $(\widetilde{S_n}, S = (s_0, s_1, \ldots, s_{n-1})$, then $\widetilde{S_n}/S_n$ is in bijection to $\widetilde{S_n}^{S \setminus s_0}$, the minimal coset representatives w.r.t S_n. In this setting, $\widetilde{S_n}/S_n$ is equipped with the Bruhat order inherited from $\widetilde{S_n}^{S \setminus s_0}$.

On the $\mathcal{C}_n$ side, we can define the following covering relation: For $p, q \in \mathcal{C}_n$, $p \succ q$ if $p = q \cup A_k(q,i)$ for some i, (or, equivalently, $q = p \setminus R_k(p,i)$). This covering relation extends to a partial ordering on $\mathcal{C}_n$.

There is another partial ordering defined on $\mathcal{C}_n$: For $p, q \in \mathcal{C}_n$, $p \supset q$ if p contains q as Young diagrams.

It is clear that $\supset$ is a stronger relation than $\succ$, that is, $p \succ q \Rightarrow p \supset q$. In fact, $\supset$ is strictly stronger than $\succ$ as demonstrated in the following example for $n = 3$: Let

$$p = sh\left(\begin{ytableau} 0 & 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 \\ 2 & 0 \\ 1 \\ 0 \end{ytableau}\right) \text{ and } q = sh\left(\begin{ytableau} 0 & 1 & 2 & 0 \\ 2 & 0 \\ 1 & 2 \\ 0 \\ 2 \end{ytableau}\right),$$ then it is clear that $p \supset q$ but $p \not\succ q$,

since there can be no such a sequence $(p = p_0, p_1, \ldots, p_k = q)$ that p_i covers p_{i+1} under $\succ$.

The following two propositions state that, under $\mathcal{C}$, $\succ$ and $\supset$ play exactly the same role as the weak (left) Bruhat order $>_L$ and the Bruhat order $>$ on $\widetilde{S_n}/S_n$, respectively.

Proposition 2.6. *$\mathcal{C}$ is an isomorphism between $(\widetilde{S_n}/S_n, >_L)$ and $(\mathcal{C}_n, \succ)$.*

Proof. First we note that by induction it suffices to show the correspondence between the covering relations from the two sides.

For the forward direction, let us assume the following inductive hypothesis: for $v \in \widetilde{S_n}/S_n$ and $l(v) = l$, if v covers w in left Bruhat order then $\mathcal{C}(v) \succ \mathcal{C}(w)$.

Now suppose $u >_L v$ and $u = s_i v$, and let $p = \mathcal{C}(u)$ and $q = \mathcal{C}(v)$, we want to conclude that $A(q,i) \neq \emptyset$. First we notice that it can not be the case that $A(q,i) = R(q,i) = \emptyset$ since this would violate the bijection between $\widetilde{S_n}/S_n$ and $\mathcal{C}_n$. Thus, if $A(q,i) = \emptyset$ we will have $R(q,i) \neq \emptyset$.

Then $p = s_i.q = q \setminus R(q,i) \prec q$. Then by induction, we should have $v \succ u$, a contradiction.

For the backward direction, if $\mathcal{C}(v) \succ \mathcal{C}(w)$ then by definition $\mathcal{C}(v) = \mathcal{C}(w) \cup A(\mathcal{C}(w), i)$ for some i, and since v and w both are the minimal coset representatives we must have $v = s_i w$. $\square$

Proposition 2.7. *$\mathcal{C}$ is an isomorphism between $(\widetilde{S_n}/S_n, >)$ and $(\mathcal{C}_n, \supset)$.*

Proof. Let us consider the following inductive hypothesis: for $v, w \in \widetilde{S_n}/S_n$ where $l(v) = l$, $v \geq w$ if and only if $p = \mathcal{C}(v) \supseteq q = \mathcal{C}(w)$.

For the base case $l = 0$, above statement is clearly true.

For $l > 0$, we know that $\mathcal{C}(v) \neq \emptyset$, thus we can pick a corner of residue i for some i. Then $s_i.\mathcal{C}(v) \succ \mathcal{C}(v)$, and by Prop 2.6 we know $s_i v <_L v$, thus $s_i v < v$. By lifting property we have

$$v \geq w \Leftrightarrow s_i v \geq \min(s_i w, w)$$

Suppose $s_i w < w$, then by induction, $\mathcal{C}(s_i v) \supseteq \mathcal{C}(s_i w)$. So we just need to show that $A(s_i w, i) \subseteq \mathcal{C}(v)$ (or equivalently, $R(w,i) \subseteq \mathcal{C}(v)$). Pick $b \in A(s_i w, i)$ and suppose that $b \notin \mathcal{C}(s_i v)$, but then it must be the case that $b \in A(s_i v, i) \subset \mathcal{C}(v)$.

Suppose $w < s_i w$, then by induction $\mathcal{C}(s_i v) \supseteq \mathcal{C}(w)$. Then we have $\mathcal{C}(v) \supset \mathcal{C}(s_i v) \supseteq \mathcal{C}(w)$.

Conversely, if $v \not\geq w$ then $s_i v \not\geq s_i w$ and $s_i v \not\geq w$, by induction this implies $\mathcal{C}(s_i v) \not\supseteq \mathcal{C}(s_i w)$ and $\mathcal{C}(s_i v) \not\supseteq \mathcal{C}(w)$. This implies that there is a corner $b \in \mathcal{C}(w)$ of residue $j \neq i$ such that $b \notin \mathcal{C}(s_i v)$. Then since $j \neq i$, $b \notin A(s_i v, i)$, thus $b \notin \mathcal{C}(v)$, that is, $\mathcal{C}(v) \not\supseteq \mathcal{C}(w)$. This finish the induction. $\square$

There is a natural bijection between standard k-tableaux to sequences in $\mathcal{C}_n$: $t \mapsto (u_0 = \emptyset, u_1, \ldots, u_m = sh(t)$, where $u_{j+1} \setminus u_j = A_k(u_j, i)$ for some i. By Prop 2.6, this sequence of i's then determines a reduced word in $\widetilde{S_n}/S_n \cong \widetilde{S_n}^{S \setminus s_0}$. The following example demonstrates this bijection:

Example 2.8. *Let $k = 2$ then from example before we know the following t is a 2-tableau*

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$$

Then the sequence of 3-cores corresponds to t is

$$\emptyset \subset sh\left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}\right) \subset sh\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}\right) \subset sh\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}\right) \subset sh\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}\right) \subset sh\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}\right) \subset sh\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}\right)$$

The reduced word determined by this sequence is $s_2 s_0 s_1 s_2 s_1 s_0$.

We summarize the above discussion to the following corollary

Corollary 2.9. *There is a bijection between $\{$standard k-tableaux$\} \leftrightarrow \{$reduced words of Grassmannian affine permutations$\}$*