LECTURE 9: LIFTING PROPERTY AND POSET STRUCTURE OF FINITE COXETER GROUPS

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1. Further Properties of the Bruhat Order

Previously we proved the following lemma:

Lemma 1. *Let $u, w \in W, u \neq w$ and $s_1 \dots s_q$ a reduced word for w, and $u = s_{i_1} \dots s_{i_k}$ is a reduced subword of $s_1 \dots s_q$. Then there exists a $v \in W$ such that:*

(1) $v > u$
(2) $l(v) = l(u) + 1$
(3) *Some reduced word of v is a subword of $s_1 \dots s_q$.*

From this, we established the *Subword Property.*

Theorem 2 (Subword Property)**.** *Let $s_1 \dots s_q$ be a reduced word for $w \in W$. Then for any $u \in W$, we have $u \leq w \iff$ there exists a reduced expression for u that is a subword of $s_1 \dots s_q$.*

From this, we observe that any interval in the Bruhat order is finite, as there are only finitely many subwords of any reduced expression. We also obtain a first result concerning automorphisms of the Bruhat order:

Corollary 3. *The map $w \mapsto w^{-1}$ is an automorphism of the Bruhat order: For any $u \leq w$, we have $u^{-1} \leq w^{-1}$.*

Proof. Given a reduced word $s_1 \dots s_q$ for w, we have $w^{-1} = s_q \dots s_1$ a reduced word for w^{-1}. Since $u \leq w$, set $u = s_{i_1} \dots s_{i_k}$ a reduced expression that is also a subword of $s_1 \dots s_q$. Then $u^{-1} = s_{i_k} \dots s_{i_1}$ is also a reduced expression, and a subword of that for w^{-1}, so the result follows from the Subword Property. □

Theorem 4 (Chain Property)**.** *Let $u < w$. Then there exists a chain $u = x_0 < x_1 < \dots x_k = w$ such that $l(x_i) = l(u) + i$.*

Proof. Follows directly from the Lemma. □

From now on let $u \lhd w$ denote a covering relation in Bruhat order; thus if $u \lhd w$ there exists no x such that $u < x < w$. In particular, by the Chain property, if $u \lhd w$, we have $l(w) = l(u) + 1$. This endows the Bruhat order with the structure of a ranked poset, with the rank of an element of W given by its length.

Proposition 5 (Lifting Property)**.** *Let $u < w$ and $s \in D_L(w)$, a left descent of w, but $s \notin D_L(u)$. Then $u \leq sw$ and $su \leq w$.*

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Proof. First notice that $s \notin D_L(u)$ implies that $u < su$. Let $sw = s_1 \dots s_q$ be a reduced word, so that $w = ss_1 \dots s_q$ is also a reduced word. Then by the subword property, there exists a reduced subword such that $u = s_{i_1} \dots s_{i_k}$. Since s is not a left descent of u, we conclude that $s_{i_1} \neq s$.

Then $u = s_{i_1} \dots s_{i_k}$ a subword of $sw = s_1 \dots s_q$, so $u < sw$. Multiply this relation on the left by s to get $su < w$. □

Definition 6. *A poset P is* directed *if for any $u, w \in P, \exists z \in P$ such that $u, w \leq z$.*

Proposition 7. *The Bruhat order is a directed poset.*

Proof. We induct on $l(u) + l(w)$.

For the base case, if $l(u) + l(w) = 0$, then $u = w = id_W$, and we can take $z = id_W$.

For $l(u) + l(w) > 0$ we can assume without loss of generality that $l(u) > 0$.

Since $u \neq id_W$, we can find an $s \in S$ such that $su < u$. By the induction hypothesis, there exists an $x \in W$ such that $su < x$ and $w < x$. We now consider two cases.

(1) $sx < x$: By the lifting property, we have $u \leq x$. But then u and w are both less than x, and we are done.
(2) $sx > x$: By the lifting property again, we have $u \leq sx$. And then $w \leq x \leq sx$, so again, we are fine.

□

2. Poset Structure of Finite Coxeter Groups

In the case of a finite Coxeter group, the directedness property implies that there exists a unique element of maximal rank, which we will denote w_0.

Proposition 8. (1) *For W finite, $\exists! w_0$ such that $w < w_0$ for all $w \in W$.*
(2) *Suppose (W, S) a Coxeter system, and there exists an $x \in W$ such that $D_L(x) = S$. Then W is finite and $x = w_0$.*

Proof. (1) The first statement follows directly from W finite and directed: Assume two maximal rank elements, then another element is greater than both, contradicting maximality.
(2) We want to show that $u \leq x$ for all $u \in W$; induct on the length of u.
For the base case, take u the identity; then $u \leq x$, as desired.
For $l(u) > 0$, there exists $s \in S$ such that $su < u$, and by induction, we have $su \leq x$. Now $s \notin D_L(u)$, and we can apply the Lifting property to get $u \leq x$, which finishes the induction.
Now $W = [id_W, x]$. Since intervals in the Bruhat order are finite, then W is finite.

□

Proposition 9. (1) $w_0^2 = id_W$
(2) $l(ww_0) = l(w_0) - l(w)$
(3) $l(w_0w) = l(w_0) - l(w)$
(4) $l(ww_0w) = l(w)$ *for all $w \in W$.*
(5) $T_L(ww_0) = T \setminus T_L(w)$
(6) $l(w_0) = |T|$

Proof.

(1) $l(w_0) = l(w_0^{-1})$. But w_0 is the unique highest rank element, so $w_0 = w_0^{-1}$, which yields the result.
(2) $l(w^{-1}) + l(ww_0) \geq l(w_0)$ by a Lemma previously proved. Notice $w_0 = w_0^{-1}$ and rearrange the inequality to get $l(ww_0) \geq l(w_0) - l(w)$. For the other inequality, we apply induction, downward from w_0. When $w = w_0$, we have $l(w_0) - l(w) = 0 = l(w_0^2)$, as desired.

   For the induction step, take $w < w_0$, and $s \in S$ such that $w < sw$. Then $l(ww_0) \leq l(sww_0) + 1 \leq l(w_0) - l(sw) + 1 = l(w_0) - l(w)$, as desired. The second inequality applies the induction hypothesis. Then we are done.
(3) For the right-sided identity, analyze the length $l((w_0w)^{-1}) = l(w^{-1}w_0)$. The result follows immediately from the previous identity.
(4) Use statement 2 twice.
(5) By 2, $tw < w \iff tww_0 > ww_0$. That is, $t \in T_L(w) \iff t \notin T_L(ww_0)$, which is exactly the statement.
(6) $l(w_0)$ is the number of elements in $T_L(w_0)$, by a previous lemma. Applying 4 with $w = id_W$ yields the result.

□

Corollary 10. *For the Bruhat order on finite Coxeter groups,*

(1) *The maps $w \to ww_0$ and $w \to w_0w$ are both anti-automorphisms of the Bruhat order.*
(2) *The map $w \to ww_0w$ is an automorphism of the Bruhat order.*

Example: Set $W = S_n$, and recall that the longest element in S_n is $w_0 = [n, n-1, \ldots, 1]$ in one-line notation. Recall also that multiplying a permutation x in one-line notation on the right by any $w \in S_n$ interchanges the places in x, while multiplying on the left interchanges the numbers.

Thus, if we take w a permutation, then ww_0 is the 'reverse' of w in one line notation. Also, we can notice that w_0w acts on the values, and sends each i in w to $n+1-i$. Conjugating w by w_0 performs both operations: reverse w and then invert its values.

For example, let $w = [4, 1, 5, 2, 3]$. Then $ww_0 = [3, 2, 5, 1, 4]$, $w_0w = [2, 5, 1, 4, 3]$, and $w_0ww_0 = [3, 4, 1, 5, 2]$.

Notice also that $w_0Sw_0 = S$, and that $x \to w_0xw_0$ is a group automorphism of S_n. This automorphism induces an automorphism of the Dynkin Diagram of S_n. In fact, there is only one automorphism of the S_n Dynkin diagram, obtained by flipping it, and this is exactly what the conjugation of S by w_0 does.

We have the following result, whose proof we omit, which shows that any automorphism of the Bruhat order which fixes the generators of W must be one of two automorphisms:

Theorem 11 (Hombergh, Waterhouse)**.** *Let (W, S) be an irreducible Coxeter system, with $|S| \geq 3$. Let $\phi : W \to W$ be an automorphism of the Bruhat order with $\phi(s) = s$ for all $s \in S$. Then $\phi(x) = x$ or $\phi(x) = x^{-1}$ for all $x \in W$.*