# Homework 1

due January 18

**Problem 1.** Show that the dominance partial order on partitions of $n$ satisfies

$$\lambda \trianglelefteq \mu \quad \iff \quad \lambda^t \trianglerighteq \mu^t,$$

where the $t$ denotes the transpose of the partition.

**Problem 2.** For $1 \leq i < j \leq n$, define the raising operator $R_{ij}$ on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).$$

(1) Show that the dominance order $\trianglelefteq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.
(2) Show that $\mu$ covers $\lambda$ if and only if $\mu = R_{ij}\lambda$, where $i, j$ satisfy the following condition: either $j = i + 1$ or $\lambda_i = \lambda_j$ (or both).
(3) Find the smallest $n$ such that the dominance order on partitions of $n$ is not a total ordering, and draw its Hasse diagram.

**Problem 3.** Let $h_i$ be the complete homogeneous symmetric functions. Show that $u_i \in \Lambda$ satisfying $u_0 = 1$ and

$$\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0 \qquad \text{for all } n \geq 1$$

are uniquely determined.

**Problem 4.** Let $w \in S_n$ be an element of the symmetric group of cycle type $\lambda$. Give a direct bijective proof that the number of elements $v \in S_n$ commuting with $w$ is equal to

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

where $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ of size $i$.

**Problem 5.** Show that

$$\prod_{\lambda \vdash n} \prod_{i \geq 1} m_i(\lambda)! = \prod_{\lambda \vdash n} \prod_{i \geq 1} i^{m_i(\lambda)}.$$