Homework 3

due February 15

Problem 1. Let $C(x, y) = \prod_{i,j \geq 1}(1 - x_i y_j)^{-1}$. Show that then for all $f \in \Lambda$

$$\langle C(x, y), f(x) \rangle = f(y)$$

where the scalar product is taken in the x variables. In other words, $C(x, y)$ is a *reproducing kernel* for the scalar product.

Problem 2. Show that

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) = \sum_{\lambda} s_\lambda(x) s_{\tilde{\lambda}^t}(y)$$

where the sum is over all partitions λ with $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$ and $\tilde{\lambda}$ is defined as

$$\tilde{\lambda} = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1).$$

Problem 3. Let $D_\mu : \Lambda \to \Lambda$ be the linear transformation given by

$$D_\mu(s_\lambda) = s_{\lambda/\mu}.$$

Show that $D_\mu D_\lambda = D_\lambda D_\mu$.

Problem 4. If R is a ring, then an additive group homomorphism $D : R \to R$ is called a derivation if $D(fg) = (Df)g + f(Dg)$ for all $f, g \in R$.

(1) Show that the linear transformation $\Lambda \to \Lambda$ defined by $D(s_\lambda) = s_{\lambda/1}$ is a derivation.
(2) Show that the bilinear operation $[f, g]$ on Λ given by

$$[s_\lambda, s_\mu] = s_{\lambda/1} s_\mu - s_\lambda s_{\mu/1}$$

defines a Lie algebra structure on Λ. That is, the Jacobi identity holds

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0.$$

Problem 5. If A is an algebra over a field k, then $A \otimes_k A$ is an algebra with multiplication characterized uniquely by $(a \otimes b)(c \otimes d) = ac \otimes bd$. A coproduct is an algebra homomorphism $\Delta : A \to A \otimes A$ which is coassociative in the sense that the two maps

$$(1 \otimes \Delta) \circ \Delta \quad \text{and} \quad (\Delta \otimes 1) \circ \Delta$$

from A to $A \otimes A \otimes A$ are equal. (This axiom is dual to the associative law for multiplication $\mu : A \otimes A \to A$, which can be formulated as $\mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$.) Here $1 : A \to A$ is the identity map.

Taking Λ to be the algebra of symmetric functions with coefficients in $k = \mathbb{Q}$ and identifying $\Lambda \otimes \Lambda$ with $\Lambda(X)\Lambda(Y)$, show that $\Delta(f) = f(X, Y)$ defines a coproduct on Λ.

Remark: An algebra equipped with a coproduct is called a bialgebra. If we also define the counit $\epsilon : \Lambda \to \mathbb{Q}$ by $\epsilon(f) = \langle f, 1 \rangle = f(0, 0, \ldots)$ and the antipode $S : \Lambda \to \Lambda$ by $Sf = f(-X)$, these together with Δ can be shown to satisfy the axioms of a Hopf algebra.

Problem 6. The relevance of this problem will become clear when we talk about differential posets in class.

(1) Let $U : \Lambda \to \Lambda$ and $D : \Lambda \to \Lambda$ be linear transformations defined by

$$U(f) = p_1 f$$
$$D(f) = \frac{\partial}{\partial p_1} f$$

where $\partial/\partial p_1$ is applied to f written as a polynomial in the p_i's. Show that

$$DU - UD = I$$
$$DU^k = kU^{k-1} + U^k D$$

where I is the identity operator.

(2) Now let us work in Young's lattice Y and define

$$D(\lambda) = \sum_{\lambda^- \lhd \lambda} \lambda^-$$
$$U(\lambda) = \sum_{\lambda^+ \rhd \lambda} \lambda^+$$

where the sum over λ^- is over all partitions being covered by λ in Y and the sum over λ^+ is over all partitions covering λ in Y. Show that

$$DU - UD = I$$
$$DU^k = kU^{k-1} + U^k D.$$

(3) Derive from (2) that

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$