Homework 5

due March 15

Problem 1. Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$, embedded as a subgroup of S_n, similarly for S_μ, and $w \in S_n$.

Problem 2. Define the Kronecker product on symmetric functions in terms of the power-sum basis by

$$p_\lambda \star p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Equivalently, the symmetric functions p_λ / z_λ are orthogonal idempotents with respect to $\star$.

(1) Prove that the Kronecker coefficients $a_{\lambda\mu\nu}$ defined by

$$s_\mu \star s_\nu = \sum_\lambda a_{\lambda\mu\nu} s_\lambda$$

are invariant under permuting the indices λ, μ, ν.

(2) Show that if $f \in \Lambda^n$, then $e_n \star f = wf$.

Remark: In fact $a_{\lambda\mu\nu}$ are non-negative integers. It is an open problem to find a combinatorial rule for the computation of the Kronecker coefficients, except for some special cases.

Problem 3. The principle specialization of a symmetric function in the variables $\{x_1, x_2, \ldots, x_m\}$ is obtained by replacing x_i by q^i for all i.

(a) Show that the Schur function specialization $s_\lambda(q, q^2, \ldots, q^m)$ is the generating function for semistandard λ-tableaux with all entries of size at most m.

(b) Define the content of cell (i, j) to be $c_{i,j} = j - i$. Prove that

$$s_\lambda(q, q^2, \ldots, q^m) = q^{m(\lambda)} \prod_{(i,j) \in \lambda} \frac{1 - q^{c_{i,j}+m}}{1 - q^{h_{i,j}}}$$

where $m(\lambda) = \sum_{i \geq 1} i\lambda_i$ and $h_{i,j}$ is the hook length of the cell (i, j) in λ.

Problem 4. Let r be a positive integer. A poset A is r-differential if it satisfies the definition from class with the second condition replaced by

- If $a \in A$ covers k elements for some k, then it is covered by $k + r$ elements.

Prove the following statements about r-differential posets A.

(a) The rank cardinalities $|A_n|$ are finite for all $n \geq 0$. (This implies that the operations D and U are well-defined).
(b) Let A be a graded poset with A_n finite for all $n \geq 0$. Then A is r-differential if and only if $DU - UD = rI$.
(c) In any r-differential poset

$$\sum_{a \in A_n} (f^a)^2 = r^n n!,$$

where f^a is the number of saturated $\emptyset - a$ chains.
(d) If A is r-differential and B is s-differential, then the product $A \times B$ is $(r + s)$-differential. So if A is 1-differential, then the r-fold product A^r is r-differential.

Problem 5. Show that the crystal operators f_i and e_i respect the Knuth relations, that is, if $w \overset{K}{\simeq} v$, then $e_i w \overset{K}{\simeq} e_i v$ (resp. $f_i w \overset{K}{\simeq} f_i v$) as long as e_i (resp. f_i) does not annihilate w. Furthermore, w and $f_i w$ have the same recording tableau under Schensted insertion. This proves in particular, that the crystal operators can be defined on semistandard tableaux.