# Homeworks

**Problem 1.** As discussed in class for positive integers $k$ and $n$, let

$$S_{k,n} = \{(X_1, \ldots, X_k) \subseteq [n]^k \mid X_1 \cap \cdots \cap X_k = \emptyset\}$$

and

$$T_{k,n} = \{(Z_1, \ldots, Z_n) \mid Z_i \subseteq [k], Z_i \neq [k]\} .$$

Show that

$$\theta \colon T_{k,n} \to S_{k,n}$$

where $i \in X_j$ if and only if $j \in Z_i$ is a bijection.

**Problem 2.** Find a combinatorial proof of

$$2^n = \sum_{k \geq 0} \binom{n}{k} \tag{1}$$

$$0 = \sum_{k \geq 0} (-1)^k \binom{n}{k}. \tag{2}$$

**Problem 3.** Find the number of solutions to $x_1 + \cdots + x_k \leq n$ into non-negative integers.

**Problem 4.** How many paths are there in the plane from $(0, 0)$ to $(m, n) \in \mathbb{N} \times \mathbb{N}$, if each step in the path is of the form $(1, 0)$ or $(0, 1)$ (i.e. unit distance due east or due north)? Give a combinatorial proof. State a higher dimensional generalization. This problem is an archetypical result in the vast subject of lattice-path counting.

**Problem 5.** Let $m, n \in \mathbb{N}$. Give a combinatorial proof of the identity $\left(\!\binom{n}{m}\!\right) = \left(\!\binom{m+1}{n-1}\!\right)$.

**Problem 6.**

(a) We defined the multinomial coefficient

$$\binom{n}{a_1, \ldots, a_m},$$

where $a_1 + \cdots + a_m = n$, to be the number of ways of assigning each element of an $n$-set to categories $C_1, \ldots, C_m$ such that category $C_i$

contains $a_i$ elements. Show that

$$\binom{n}{a_1, \ldots, a_m} = \frac{n!}{a_1! \cdots a_m!}\ .$$

(b) Show that $\binom{n}{a_1,\ldots,a_m}$ is the coefficient of $x_1^{a_1} \cdots x_m^{a_m}$ in the expansion of $(x_1 + \cdots + x_m)^n$.

**Problem 7.** Read the third proof of Proposition 1.3.7 on page 27/28 of Stanley's book and fill in the details for the bijection.

**Problem 8.** Prove the convolution formula for the binomial coefficients

$$\sum_j \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$$

in two ways:

(1) with generating functions;
(2) directly combinatorially.

**Problem 9.** Prove that the $q$-binomial coefficients satisfy the initial condition $\begin{bmatrix} 0 \\ k \end{bmatrix} = \delta_{k,0}$ for $k \geq 0$ and the recursion relation for $0 \leq k \leq n$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\ .$$

**Problem 10.** Show that

$$\frac{1}{(q)_\infty} = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k^2}$$

using partitions. (*Hint:* Use the Durfee square of a partition $\lambda$ which is the largest square $(i^i)$ contained in $\lambda$).

**Problem 11.** Let $p(\mathcal{O}, n)$ be the number of partitions of $n$ into odd parts and let $p(\mathcal{D}, n)$ be the number of partitions of $n$ into distinct parts. Prove that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$.

**Problem 12.** Give a combinatorial proof of the following $q$-analog of the convolution formula for binomial coefficients

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_{i+j=k} q^{(m-i)j}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix}.$$

**Problem 13.**

(1) Prove the $q$-binomial theorem

$$\prod_{i=0}^{m-1}(1+xq^i)=\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}q^{\binom{j}{2}}x^j,$$

where $m$ is a nonnegative integer and $x$ is an indeterminate.

(2) Deduce from this

$$\prod_{i=1}^{s}(1+x^{-1}q^i)\prod_{i=0}^{t-1}(1+xq^i)=\sum_{j=-s}^{t}\begin{bmatrix}s+t\\s+j\end{bmatrix}q^{\binom{j}{2}}x^j.$$

(3) By letting $s$ and $t$ tend to infinity, prove Jacobi's triple product identity:

$$\sum_{j\in\mathbb{Z}}(-1)^jq^{\binom{j}{2}}x^j=\prod_{i\geq 0}(1-xq^i)(1-x^{-1}q^{i+1})(1-q^{i+1}).$$

**Problem 14.** From the formula for $p_e(\mathcal{D},n)-p_o(\mathcal{D},n)$, prove Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{m=-\infty}^{\infty}(-1)^mq^{\frac{1}{2}m(3m-1)}\ .$$

**Problem 15.** The famous first Rogers–Ramanujan identity is given by

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+1})(1-q^{5j+4})},$$

where $(q)_n=(1-q)(1-q^2)\cdots(1-q^n)$ for $n>0$ and $(q)_0=1$.

(1) Show that the right-hand side is the generating function of partitions with parts congruent $1$ or $4$ modulo $5$.

(2) Consider $L+1$ points on a line labeled by $i=0,1,2,\dots,L$. Assign to each point a height variable $\sigma_i$ which takes on the values $0$ or $1$. In addition the height variables satisfy the restrictions $\sigma_0=\sigma_L=0$ and $\sigma_i\sigma_{i+1}=0$. An allowed configuration of height variables for a given length $L$ is called a *path* of length $L$. One can illustrate a path graphically by drawing all points $(i,\sigma_i)$ and connecting adjacent points by straight lines. An example for a path with $L=9$ is given in figure 1. The condition $\sigma_i\sigma_{i+1}=0$ requires that the paths consist of a certain number of non-overlapping triangles. To each path $p$

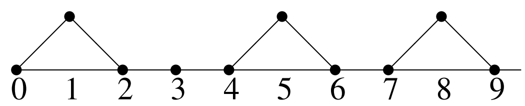


FIGURE 1. A path of length $9$

one may assign an energy $E(p)$ by summing up the positions of the peaks, that is

$$E(p) = \sum_{j=1}^{L} j\sigma_j.$$

Define the function

$$F(L) = \sum_{p} q^{E(p)}$$

where the sum is over all paths of length $L$. Show that $F(L)$ satisfies the initial condition $F(0) = F(1) = 1$ and recursion relation

$$F(L) = F(L-1) + q^{L-1}F(L-2).$$

(3) Prove that

$$F(L) = \sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix}.$$

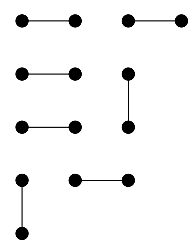
(4) Show that $\lim_{L\to\infty} F(L)$ yields the left-hand side of the Rogers–Ramanujan identity.

(5) Use the path interpretation to show that the left-hand side of the Rogers–Ramanujan identity is the generating function of partitions for which the difference between any two parts is at least two.

(6) Conclude that the Rogers–Ramanujan identity implies that the number of partitions of an integer $N$ into parts in which the difference between any two parts is at least 2 is the same as the number of partitions of $N$ into parts congruent to $1$ or $4$ modulo $5$.
Verify this statement for $N = 6$.

**Problem 16.** Show that the number $s(n)$ of partitions of $n$ that are self-conjugate ($\lambda = \lambda^t$) is equal to the number of partitions of $n$ into distinct odd parts. What is the generating function $F(x) = \sum_{n\geq 0} s(n)x^n$?

**Problem 17.** Let $f(n)$ be the number of partitions of $2n$ whose Ferrers diagram can be covered by $n$ edges, each connecting two adjacent dots. For instance, $(4,3,3,3,1)$ can be covered as follows:



Show that

$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1} \frac{1}{(1-x^i)^2}.$$

(Hint: This can be shown combinatorially. Do you recognize the right-hand side? Can you find a bijection with the left?)

**Problem 18.** Give a simple "balls into boxes" proof that the total number of parts of all compositions of $n$ is equal to $(n+1)2^{n-2}$. (The simplest argument expresses the answer as a sum of two terms.)

**Problem 19.** (difficult) Prove the Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})},$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n>0$ and $(q)_0 = 1$.

**Problem 20.** Let $S = \{P_1, \ldots, P_n\}$ be a set of properties, and let $f_k$ (resp. $f_{\geq k}$) denote the number of objects in a finite set $A$ that have exactly $k$ (resp. at least $k$ of the) properties. Show that

$$f_k = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} g_i,$$

and

$$f_{\geq k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i-1}{k-1} g_i,$$

where

$$g_i = \sum_{T\subseteq S, |T|=i} f_{\geq}(T).$$

**Problem 21.** Recall that $D(n)$ is the number of derangement in $\mathfrak{S}_n$. Give a direct combinatorial proof of the recursion

$$D(n) = nD(n-1) + (-1)^n.$$

**Problem 22.** Give a sieve-theoretic proof of the pentagonal number formula

$$1 = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} .$$

Your sieve should start with all partitions of $n \geq 0$ and sieve out all but the empty partition of $0$.

**Problem 23.** Let $P$ be a locally finite poset and $I(P)$ the incidence algebra of $P$ over $\mathbb{C}$. Define $\eta \in I(P)$ as

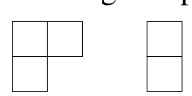
$$\eta(x,y) = \begin{cases} 1 & \text{if } y \text{ covers } x \\ 0 & \text{else} \end{cases}$$

for all $x \leq y$. Show that $(1-\eta)^{-1}(x,y)$ is equal to the total number of maximal chains in $[x,y]$.

**Problem 24.** Recall that $D_n$ is the set of all positive integral divisors of $n$ with the order that $r \leq s$ in $D_n$ if $s$ is divisible by $r$, denoted $r \mid s$.

(1) Derive a formula for the Möbius function $\mu(r,s)$ for the poset $D_n$.
(2) Show that $\sum_{d|n} \mu(1,d) = 0$.

**Problem 25.** Let $f(n)$ be the number of ways a $2 \times n$ chessboard can be partitioned into copies of the following two pieces:



Any rotation or reflection of the pieces is allowed. For example, $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 5$. Find an explicit expression for $F(x) = \sum_{n \geq 0} f(n)x^n$.

**Problem 26.** Show that the Catalan number $C_n$ is equal to the number of standard Young tableaux of shape $(n, n)$.

**Problem 27.** Consider words $w$ consisting of $n+1$ $L$'s and $n$ $R$'s beginning with $L$, so that the number of such words is $\binom{2n}{n}$. A rotation of $w$ is a word of the form $YX$ where $w = XY$.

(1) Prove that the $2n + 1$ rotations of each word $w$ as above are all distinct.
(2) Prove that among the $n + 1$ rotations of $w$ starting with $L$ exactly one is of the form $LX$, where $X$ is a Dyck path. This gives a combinatorial interpretation of the formula for $C_n$.

**Problem 28.** Show that the Catalan number $C_n$ is the number of all peaks of height one in all Dyck paths from $(0, 0)$ to $(2n, 0)$.

**Problem 29.** Let $f$ and $g$ be functions on a finite lattice $L$ satisfying

$$f(x) = \sum_{\substack{y \\ x \wedge y = \widehat{0}}} g(y). \tag{3}$$

Show that if $\mu(\widehat{0}, x) \neq 0$ for all $x \in L$, then (3) can be inverted to yield

$$g(x) = \sum_y \alpha(x, y) f(y),$$

where

$$\alpha(x, y) = \sum_t \frac{\mu(x, t)\mu(y, t)}{\mu(\widehat{0}, t)}.$$

**Problem 30.** Prove that the sliding definition of promotion $\partial$ on a finite poset due to Schützenberger is equivalent to the definition

$$\partial = \tau_1 \tau_2 \cdots \tau_{n-1}$$

acting on the right (i.e. $\tau_1$ acts first, then $\tau_2$ etc). Here $\tau_i$ acts on a poset $P$ by interchanging $i$ and $i + 1$ if $i$ and $i + 1$ are incomparable in $P$ and otherwise acts as the identity.

**Problem 31.** Prove that the order of the promotion operator on rectangular standard tableaux with $n$ boxes divides $n$.

**Problem 32.** Tile $\mathbb{R}^2$ by lines with slope 1 and $-1$ through the integer points $(n, 0)$ on the $x$-axis. The area of a Dyck path is defined to be the number of full squares in this tiling under the Dyck path and above the $x$-axis. Define the bounce path of a Dyck path of length $2n$ by starting at point $(2n, 0)$ and following the Dyck path up from right to left. Whenever the Dyck path goes down from right to left, the bounce path will also go down all the way to the $x$-axis and then move back up until it hits the Dyck path again, follow it up, until the Dyck path turns down again. At this point the bounce path also turns down and continues to go down until it hits the $x$-axis etc. The bounce statistic of a Dyck path is the sum of the positions where the bounce path meets the $x$-axis. Give a bijection on Dyck paths $D_n$ of length $2n$ to $D_n$ which interchanges the area and bounce statistic.

**Problem 33.** Let $G_n$ be the graph with vertex set $V = [n]$ and edge set

$$E = \{12, 13, 14, \ldots, 1n, 23\}.$$

Find the number of spanning trees of $G_n$ in two ways: by a direct count and by using the Matrix-Tree Theorem.

**Problem 34.** The complete bipartite graph, $K_{m,n}$, has vertex set $V = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ and edge set consisting of $v_i w_j$ for all $i, j$ (and no other edges). Show that

$$\#\mathcal{ST}(K_{m,n}) = m^{n-1} n^{m-1}.$$

**Problem 35.** Let $L$ be a lattice with meet $\wedge$ and join $\vee$. Show that:

(1) $\wedge$, $\vee$ are associate, commutative and idempotent ($x \wedge x = x \vee x = x$).
(2) The absorption law holds $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.
(3) $x \wedge y = x$ if and only if $x \vee y = y$ if and only if $x \leq y$.
(4) All finite lattices have $\hat{0}$ and $\hat{1}$.