SOME PARTICULAR ENTRIES OF THE TWO-PARAMETER KOSTKA MATRIX

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Recall the definition of the two-parameter Kostka Matrix, defined in Macdonald's paper [4]:

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t)S_\lambda(x;t).$$

It is now known that each $K_{\lambda,\mu}(q,t)$ is a polynomial in q and t with positive integral coefficients, though it's hard enough just to prove that they are polynomials, and no nice combinatorial interpretation is known in general. We will see the statements and summaries of short, elementary proofs of the following:

(1) An explicit formula for $K_{\lambda,\mu}(q,t)$ for partitions μ with at most two columns or, equivalently (by duality), at most two rows.

(2) A combinatorial interpretation of $K_{\lambda,\mu}(q,t)$ for the case in which μ is a hook.

For complete proofs, see Stembridge's paper [5].

For the statement of the first theorem, we need the Gaussian polynomial (or q-binomial coefficient) $\begin{bmatrix} r \\ s \end{bmatrix}_t = \frac{(t^r;t^{-1})_s}{(t^s;t^{-1})_s}$. Recall (or be amazed!) that this gives us a generating function for partitions whose diagrams fit in an $(r-s)\times s$ box, and counts the number of s-dimensional subspaces of an r-dimensional vector space over a field of t elements.

Theorem 1. We have

$$K_{\lambda,2^r1^{n-2r}}(q,t) = \sum_{s=0}^{r} q^{r-s}(t^{n-r}q;t^{-1})_s \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda,2^s1^{n-2s}}(t).$$

For the proof, we define $a_{\lambda,\mu}(q,t)$ to be the transition matrix between $J_\mu(q,t)$ and the Hall-Littlewood functions $Q_\lambda(t)$, i.e.

$$J_\mu(q,t) = \sum_\lambda a_{\lambda,\mu}(q,t)Q_\lambda(t).$$

This matrix is upper-triangular. We also have the Kostka-Foulkes matrix, defined by

$$Q_\mu(t) = \sum_\lambda K_{\lambda,\mu}(t)S_\lambda.$$

Since $J_\mu(0,t) = Q_\mu(t)$, we can derive

$$K_{\lambda,\mu}(q,t) = \sum_\nu K_{\lambda,\nu}(t)a_{\nu,\mu}(q,t).$$

For $0 \le r,s \le n/2$, we define $A_{s,r}(q,t) = a_{2^s1^{n-2s},2^r1^{n-2r}}(q,t)$ for convenience. Then it suffices to prove

$$A_{s,r}(q,t) = q^{r-s}(t^{n-r}q;t^{-1})_s \begin{bmatrix} r \\ s \end{bmatrix}_t.$$

For this, we bring in the specialization homomorphism $\epsilon_{a,t}$. We know [4] that

$$\epsilon_{a,t}J_\mu(q,t) = \prod_{(i,j)\in\lambda}(t^{i-1}-q^{j-1}a).$$

In the special case $\mu = 2^r 1^{n-2r}$, we get

$$\epsilon_{a,t} J_\mu(q,t) = \prod_{i=1}^{n-r} (t^{i-1} - a) \prod_{i=1}^{r} (t^{i-1} - qa)$$

$$\epsilon_{a,t} Q_\mu(t) = t^{\binom{r}{2}} \prod_{i=1}^{n-r} (t^{i-1} - a).$$

The rest of the proof is basically just calculation. After a few lines, we observe that $A_{s,r}$ is a transition matrix between two bases of the polynomial ring $F[a]$, where $F = \mathbf{Q}(q,t)$. The problem then reduces to proving one horrible-looking equation, which (after a few notational changes) follows easily from the q-binomial theorem.

Remarks. The proof allows us to give an explicit formula for $J_{2^r 1^{n-2r}}(q,t)$, and to conclude that $K_{\lambda,\mu}(q,t) \in \mathbf{Z}[q,t]$ iff $a_{\nu,\mu}(q,t) \in \mathbf{Z}[q,t]$ for all $\nu \leq \mu$. Susanna Fishel has given a combinatorial formula in the two-column case using Theorem 1 and rigged configurations [1]. Also, L. Lapointe and J. Morse give a combinatorial formula in the two-column case [2].

For the hook case, we start with some definitions. Let λ be a partition of n, and let $SYT(\lambda)$ denote the set of standard Young tableaux of shape λ. For any $T \in SYT(\lambda)$, let $D(T)$ denote the *descent set* of T, i.e. the subset of $\{1,\ldots,n-1\}$ with $i \in D(T)$ iff i appears in a row strictly lower (in French notation) than $i+1$ in T. For example, if T is

$$\begin{array}{cc} 5 & \\ 3 & 4 \\ 1 & 2 \end{array}$$

then $D(T) = \{2,4\}$. As an exercise, work out the descent sets for the other elements of $SYT(221)$.

For each $r \in \mathbf{N}$, we also define

$$\alpha_r(T) = \sum_{1 \le k < r,\ k \in D(T)} k, \qquad \beta_r(T) = \sum_{r \le k < n,\ k \notin D(T)} n - k.$$

Theorem 2. We have

$$K_{\lambda, r1^{n-r}(q,t)}(q,t) = \sum_{T \in SYT(\lambda)} q^{\alpha_r(T)} t^{\beta_r(T)}.$$

Exercise: Compute $K_{221,311}(q,t)$.

The idea of the proof is to look at $J_r \cdot J_{1^s}$ in two different ways: one in terms of S_λ's, and the other in terms of other J's. The computation involves several references to both [3] and [4].

References

[1] S. Fishel, *Statistics for special q,t-Kostka polynomials*, Proc. Amer. Math. Soc. **123** (1995), pp. 2961–2969.

[2] L. Lapointe and J. Morse, *Tableaux statistics for two part Macdonald polynomials*, math.CO/9812001.

[3] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, Oxford, 1979.

[4] I. G. Macdonald, *A new class of symmetric functions*, Publ. I.R.M.A. Strasbourg, Actes 20^e Séminaire Lotharingien, 1988, pp. 131–171.

[5] J. R. Stembridge, *Some particular entries of the two-parameter Kostka matrix*, Proc. Amer. Math. Soc. **121** (1994), pp. 367–373.