SUPPLEMENTARY NOTES ON "A BIJECTION BETWEEN TYPE $D_n^{(1)}$ CRYSTALS AND RIGGED CONFIGURATIONS"

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These notes supplement [2, Appendix C].

1. Proof of $[\delta, \tilde{\delta}] = 0$

One may easily verify that (see also [1, Eq. (3.10)])

$$\begin{aligned} -p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} &= -\sum_{b\in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)} + L_i^a \\ &\geq -\sum_{b\in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)}, \end{aligned} \tag{1.1}$$

The proof of $[\delta, \tilde{\delta}] = 0$ is given here by Lemmas 1.1 and 1.2 below. We rely here heavily on [1, Appendix A].

Let $(\nu, J) \in \mathrm{RC}(\lambda, B)$ where $B = (B^{1,1})^{\otimes 2} \otimes B'$. The following notation is used:

$$\begin{aligned} \delta(\nu, J) &= (\dot{\nu}, \dot{J}) \\ \tilde{\delta}(\nu, J) &= (\tilde{\nu}, \tilde{J}) \\ \tilde{\delta}\circ\delta(\nu, J) &= (\tilde{\dot{\nu}}, \tilde{\dot{J}}) \\ \delta\circ\tilde{\delta}(\nu, J) &= (\dot{\tilde{\nu}}, \dot{\tilde{J}}). \end{aligned}$$

Furthermore, let $\{\dot{\ell}^{(k)}, \dot{s}^{(k)}\}$, $\{\tilde{\ell}^{(k)}, \tilde{s}^{(k)}\}$, $\{\tilde{\dot{\ell}}^{(k)}, \tilde{\dot{s}}^{(k)}\}$ and $\{\dot{\tilde{\ell}}^{(k)}, \dot{\tilde{s}}^{(k)}\}$ be the lengths of the strings that are shortened in the transformations $(\nu, J) \mapsto (\dot{\nu}, \dot{J})$, $(\nu, J) \mapsto (\tilde{\nu}, \tilde{J})$, $(\dot{\nu}, \dot{J}) \mapsto (\tilde{\dot{\nu}}, \tilde{\dot{J}})$ and $(\tilde{\nu}, \tilde{J}) \mapsto (\dot{\tilde{\nu}}, \dot{\tilde{J}})$, respectively. We call the strings, whose lengths are labeled by an ℓ, ℓ-strings and those labeled by an s, s-strings.

Lemma 1.1. *The following cases occur at $(\nu, J)^{(k)}$:*

I. ***Nontwisted case.*** *In this case the ℓ-string selected by δ (resp. $\tilde{\delta}$) in $(\nu, J)^{(k)}$ is different from the s-string selected by $\tilde{\delta}$ (resp. δ) in $(\nu, J)^{(k)}$. For the ℓ-strings one of the following must hold:*

(ℓa) ***Generic case.*** *If δ and $\tilde{\delta}$ do not select the same ℓ-string, then $\dot{\tilde{\ell}}^{(k)} = \dot{\ell}^{(k)}$ and $\tilde{\dot{\ell}}^{(k)} = \tilde{\ell}^{(k)}$.*

(ℓb) ***Doubly singular case.*** *In this case δ and $\tilde{\delta}$ select the same ℓ-string, so that $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$. Then*

(1) *If $\tilde{\dot{\ell}}^{(k)} < \ell$ (or $\dot{\tilde{\ell}}^{(k)} < \ell$) then $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell - 1$ and $m_{\ell-1}^{(k+1)} = 0$ for $k < n-2$, $m_{\ell-1}^{(n-1)} = m_{\ell-1}^{(n)} = 0$ for $k = n-2$ and $m_{\ell-1}^{(n-2)} = 0$ for $k = n-1, n$.*

(2) *If $\tilde{\dot{\ell}}^{(k)} = \ell$ (or $\dot{\tilde{\ell}}^{(k)} = \ell$) then case I.($\ell$s)(1') (or I.($\ell$s)(1)) holds or $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell$.*

(3) *If* $\tilde{\dot{\ell}}^{(k)} > \ell$ *(or* $\dot{\tilde{\ell}}^{(k)} > \ell$*) then case I.(*$\ell s$*)(1',2) (or I.(*$\ell s$*)(1,2)) holds or* $\dot{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ *and* $\tilde{\dot{\ell}}^{(k)} \leq \tilde{\ell}^{(k+1)}$, $\dot{\tilde{\ell}}^{(k)} \leq \dot{\ell}^{(k+1)}$ *for* $k < n-2$, $\tilde{\dot{\ell}}^{(n-2)} \leq \min\{\tilde{\ell}^{(n-1)}, \tilde{\ell}^{(n)}\}$, $\dot{\tilde{\ell}}^{(n-2)} \leq \min\{\dot{\ell}^{(n-1)}, \dot{\ell}^{(n)}\}$ *for* $k = n-2$*, and* $\tilde{\dot{\ell}}^{(k)} \leq \tilde{s}^{(n-2)}$, $\dot{\tilde{\ell}}^{(k)} \leq \dot{s}^{(n-2)}$ *for* $k = n-1, n$.

For the s*-strings, case I.(*ℓs*) holds or one the following must hold:*

(sa) ***Generic case.*** *If* δ *and* $\tilde{\delta}$ *do not select the same* s*-string, then* $\dot{\tilde{s}}^{(k)} = \dot{s}^{(k)}$ *and* $\tilde{\dot{s}}^{(k)} = \tilde{s}^{(k)}$.

(sb) ***Doubly singular case.*** *In this case* δ *and* $\tilde{\delta}$ *select the same* s*-string, so that* $\dot{s}^{(k)} = \tilde{s}^{(k)} =: s$*. Then*

(1) *If* $\tilde{\dot{s}}^{(k)} < s$ *(or* $\dot{\tilde{s}}^{(k)} < s$*) then* $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = s-1$ *and* $m_{s-1}^{(k-1)} = 0$.

(2) *If* $\tilde{\dot{s}}^{(k)} = s$ *(or* $\dot{\tilde{s}}^{(k)} = s$*) then* $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = s$.

(3) *If* $\tilde{\dot{s}}^{(k)} > s$ *(or* $\dot{\tilde{s}}^{(k)} > s$*) then* $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)}$, $\tilde{\dot{s}}^{(k)} \leq \tilde{s}^{(k-1)}$ *and* $\dot{\tilde{s}}^{(k)} \leq \dot{s}^{(k-1)}$.

(ℓs) ***Mixed case.*** *One of the following holds:*

(1) $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\tilde{\dot{\ell}}^{(k)} = \dot{s}^{(k)} = \dot{\ell}^{(k+1)} =: \ell'$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} =: \ell''$, $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$ *or possibly the same conditions for* ℓ *and* ℓ', $\dot{\tilde{\ell}}^{(k)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \ell''$, $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell'''$, $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k)} = 1$, $m_{\ell''}^{(k+1)} = 2$ *if case I.(*ℓs*)(1) does not hold at* $k-1$*. Furthermore, either* $\tilde{\dot{\ell}}^{(k)} \leq \tilde{\ell}^{(k+1)}$ *or case I.(*ℓs*)(1) holds at* $k+1$ *with the same values of* ℓ' *and* ℓ''*. Similarly, either* $\dot{\tilde{s}}^{(k)} \leq \dot{s}^{(k-1)}$ *or case I.(*ℓs*)(1) holds at* $k-1$ *with the same values of* ℓ' *and* ℓ''.

(1') $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\dot{\tilde{\ell}}^{(k)} = \tilde{s}^{(k)} = \tilde{\ell}^{(k+1)} =: \ell'$, $\tilde{\dot{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} =: \ell''$, $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ *or possibly the same conditions for* ℓ *and* ℓ', $\tilde{\dot{\ell}}^{(k)} = \dot{s}^{(k)} = \dot{s}^{(k+1)} = \ell''$, $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell'''$, $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k)} = 1$, $m_{\ell''}^{(k+1)} = 2$ *if case I.(*ℓs*)(1') does not hold at* $k-1$*. Furthermore, either* $\dot{\tilde{\ell}}^{(k)} \leq \dot{\ell}^{(k+1)}$ *or case I.(*ℓs*)(1') holds at* $k+1$ *with the same values of* ℓ' *and* ℓ''*. Similarly, either* $\tilde{\dot{s}}^{(k)} \leq \tilde{s}^{(k-1)}$ *or case I.(*ℓs*)(1') holds at* $k-1$ *with the same values of* ℓ' *and* ℓ''.

(2) *For* $k < n-2$ *(resp.* $k = n-2$*)* $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\dot{s}^{(k)} = \tilde{s}^{(k)} = \dot{s}^{(k+1)} = \tilde{s}^{(k+1)} =: \ell'$ *(resp.* $\dot{s}^{(k)} = \tilde{s}^{(k)} = \dot{\ell}^{(n-1)} = \dot{\ell}^{(n)} = \tilde{\ell}^{(n-1)} = \tilde{\ell}^{(n)} = \ell'$*),* $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k)} = \ell''$, $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} := \ell'''$ *and case I.(*ℓs*)(2) holds at* $k+1$ *(resp.* $n-1$ *and* n*) with the same values of* ℓ' *and* ℓ'' *and* $\ell = \ell'$, $\ell''' = \ell''$*. Also, either* $\tilde{\dot{s}}^{(k)} \leq \tilde{s}^{(k-1)}$ *and* $\dot{\tilde{s}}^{(k)} \leq \dot{s}^{(k-1)}$ *or case I.(*ℓs*)(2) holds at* $k-1$ *with the same values of* ℓ' *and* ℓ''*. For* $k = n-1, n$, $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \dot{s}^{(n-2)} = \tilde{s}^{(n-2)} = \ell'$ *and* $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell''$*. In addition case I.(*$\ell s$*)(2) holds at* $n-2$ *with the same values of* ℓ' *and* ℓ''.

II. ***Twisted case.*** *In this case the* ℓ*-string in* $(\nu, J)^{(k)}$ *selected by* δ *is the same as the* s*-string selected by* $\tilde{\delta}$ *or vice versa. In the first case* $\dot{\ell}^{(k)} = \tilde{s}^{(k)} =: \ell$*. Then* $\tilde{\ell}^{(k)} = \tilde{\dot{\ell}}^{(k)}$ *and one of the following holds:*

(1) *If* $\dot{\tilde{\ell}}^{(k)} < \ell$*, then* $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell - 1$, $m_{\ell-1}^{(k+1)} = 0$ *or* $m_{\ell-1}^{(k+1)}(\tilde{\nu}) = 0$*, and* $m_{\ell-1}^{(k-1)} = 0$ *or* $m_{\ell-1}^{(k-1)}(\dot{\nu}) = 0$*. Furthermore* $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$.

(2) *If* $\dot{\tilde{\ell}}^{(k)} = \ell$*, then* $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell$ *and* $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$.

(3) *If* $\dot{\tilde{\ell}}^{(k)} > \ell$*, then*

(i) $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)}$ *and* $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$*, or*

(ii) $\dot{\tilde{\ell}}^{(k)} = \dot{s}^{(k)}$ *and* $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} \leq \dot{s}^{(k-1)}$.

Furthermore, either $\dot{\tilde{\ell}}^{(k)} \leq \dot{\ell}^{(k+1)}$ *or* $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)}$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k+1)}$, $m_\ell^{(k+1)} = 1$ *and Case II.(3)(i) holds at* $k+1$*. Similarly, either* $\tilde{\dot{s}}^{(k)} \leq \tilde{s}^{(k-1)}$ *or* $\dot{\ell}^{(k)} = \dot{\ell}^{(k-1)}$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k-1)}$, $m_\ell^{(k-1)} = 1$ *and Case II.(3) holds at* $k-1$.

If the ℓ*-string in* $(\nu, J)^{(k)}$ *selected by* $\tilde{\delta}$ *is the same as the* s*-string selected by* δ*, then* $\tilde{\ell}^{(k)} = \dot{s}^{(k)} =: \ell$*. In this case* $\dot{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ *and one of the following holds:*

(1') *If* $\tilde{\dot{\ell}}^{(k)} < \ell$*, then* $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell - 1$, $m_{\ell-1}^{(k+1)} = 0$ *or* $m_{\ell-1}^{(k+1)}(\dot{\nu}) = 0$*, and* $m_{\ell-1}^{(k-1)} = 0$ *or* $m_{\ell-1}^{(k-1)}(\tilde{\nu}) = 0$*. Furthermore* $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$.

(2') *If* $\tilde{\dot{\ell}}^{(k)} = \ell$*, then* $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell$ *and* $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$.

(3') *If* $\tilde{\dot{\ell}}^{(k)} > \ell$*, then*

(i) $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)}$ *and* $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$*, or*

(ii) $\tilde{\dot{\ell}}^{(k)} = \tilde{s}^{(k)}$ *and* $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} \leq \tilde{\ell}^{(k-1)}$.

Furthermore, either $\tilde{\dot{\ell}}^{(k)} \leq \tilde{\ell}^{(k+1)}$ *or* $\tilde{\ell}^{(k)} = \tilde{\ell}^{(k+1)}$, $\tilde{\dot{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k+1)}$, $m_\ell^{(k+1)} = 1$ *and Case II.(3')(i) holds at* $k+1$*. Similarly, either* $\dot{\tilde{s}}^{(k)} \leq \dot{s}^{(k-1)}$ *or* $\tilde{\ell}^{(k)} = \tilde{\ell}^{(k-1)}$, $\tilde{\dot{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k-1)}$, $m_\ell^{(k-1)} = 1$ *and Case II.(3') holds at* $k-1$.

Lemma 1.2. $\tilde{\dot{J}} = \dot{\tilde{J}}$.

Proof of Lemma 1.1. The proof proceeds by induction on k in the following way. For $k = 0, 1, 2, \ldots, n$ the statements about the ℓ-strings are proved assuming that the statements about the ℓ-strings hold for $i = 1, 2, \ldots, k-1$. The statements about the s-strings are proved by induction on $k = n-2, n-3, \ldots, 1$ assuming that the statements for all ℓ-strings and the s-strings for $i = n-2, n-3, \ldots, k+1$ hold.

For the base case $k = 0$ we have $\dot{\ell}^{(0)} = \tilde{\ell}^{(0)} = \tilde{\dot{\ell}}^{(0)} = \dot{\tilde{\ell}}^{(0)} = 1$.

Note that

$$
(1.2) \qquad \begin{aligned} \tilde{\dot{\ell}}^{(k)} &\leq \tilde{\ell}^{(k+1)} \\ \dot{\tilde{\ell}}^{(k)} &\leq \dot{\ell}^{(k+1)} \end{aligned} \qquad \text{for } 1 \leq k < n-2, \qquad \begin{aligned} \tilde{\dot{\ell}}^{(n-2)} &\leq \min\{\tilde{\ell}^{(n-1)}, \tilde{\ell}^{(n)}\} \\ \dot{\tilde{\ell}}^{(n-2)} &\leq \min\{\dot{\ell}^{(n-1)}, \dot{\ell}^{(n)}\} \end{aligned}
$$

unless case I.(ℓs)(1),(1'),(2) or II.(3),(3') holds at k and $k+1$. Similarly,

$$
(1.3) \qquad \begin{aligned} \tilde{\dot{s}}^{(k)} &\leq \tilde{s}^{(k-1)} \\ \dot{\tilde{s}}^{(k)} &\leq \dot{s}^{(k-1)} \end{aligned} \qquad \text{for } 1 < k \leq n-2, \qquad \begin{aligned} \max\{\tilde{\dot{\ell}}^{(n-1)}, \tilde{\dot{\ell}}^{(n)}\} &\leq \tilde{s}^{(n-2)} \\ \max\{\dot{\tilde{\ell}}^{(n-1)}, \dot{\tilde{\ell}}^{(n)}\} &\leq \dot{s}^{(n-2)} \end{aligned}
$$

unless case I.(ℓs)(1),(1'),(2) or II.(3),(3') holds at k and $k-1$.

I. Nontwisted case. For this case many arguments go through as in the proof for type A as in [1, Appendix A]. Here we mainly point out the differences.

Case (ℓa). The proof of the generic case is very similar to the proof of the generic case for type A [1, Appendix A]. We focus here on $k \leq n-2$. Observe that $\tilde{\ell}^{(k)} = \tilde{\dot{\ell}}^{(k)}$ is obtained from $\dot{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ by the involution θ. Hence we only prove the latter. The singular string in $(\nu, J)^{(k)}$ of length $\dot{\ell}^{(k)}$ remains singular in passing to $(\tilde{\nu}, \tilde{J})^{(k)}$. Since $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ by (1.2), it follows that $\dot{\tilde{\ell}}^{(k)} \leq \dot{\ell}^{(k)}$.

If $\dot{\tilde{\ell}}^{(k)} = \dot{\ell}^{(k)}$ we are done. By induction hypothesis, $\dot{\tilde{\ell}}^{(k)} \geq \dot{\tilde{\ell}}^{(k-1)} \geq \dot{\ell}^{(k-1)} - 1$. If $\dot{\ell}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} < \dot{\ell}^{(k)}$, this is only possible if the string selected by δ acting on $(\tilde{\nu}, \tilde{J})^{(k)}$ is a string shortened by $\tilde{\delta}$ acting on $(\nu, J)^{(k)}$. This string in $(\tilde{\nu}, \tilde{J})^{(k)}$ has length either $\tilde{\ell}^{(k)} - 1$ or $\tilde{s}^{(k)} - 1$ and label 0. We show that this cannot occur. For this it suffices to show that

$$(1.4) \qquad p^{(k)}_{\tilde{\ell}^{(k)}-1}(\tilde{\nu}) > 0 \qquad \text{if } \dot{\ell}^{(k-1)} < \tilde{\ell}^{(k)} \le \dot{\ell}^{(k)} \text{ and } \dot{\tilde{\ell}}^{(k-1)} < \tilde{\ell}^{(k)}$$

$$(1.5) \qquad p^{(k)}_{\tilde{s}^{(k)}-1}(\tilde{\nu}) > 0 \qquad \text{if } \dot{\ell}^{(k-1)} < \tilde{s}^{(k)} \le \dot{\ell}^{(k)} \text{ and } \dot{\tilde{\ell}}^{(k-1)} < \tilde{s}^{(k)}.$$

If $\dot{\ell}^{(k-1)} - 1 = \dot{\tilde{\ell}}^{(k)} < \dot{\ell}^{(k)}$, case I.($\ell$b)(1) or II.(1) occurs at $k-1$, so that $m^{(k)}_{\dot{\ell}^{(k-1)}-1} = 0$ or $m^{(k)}_{\dot{\ell}^{(k-1)}-1}(\tilde{\nu}) = 0$. Hence $\dot{\tilde{\ell}}^{(k)} = \dot{\ell}^{(k-1)} - 1$ can only occur if $\tilde{\ell}^{(k)} = \tilde{\ell}^{(k-1)} = \dot{\ell}^{(k-1)}$ if case I.(ℓb)(1) holds at $k-1$ or $\tilde{s}^{(k)} = \tilde{s}^{(k-1)} = \dot{\ell}^{(k-1)}$ if case II.(1) holds at $k-1$. To prove that this cannot happen it suffices to show that

$$(1.6) \qquad p^{(k)}_{\tilde{\ell}^{(k)}-1}(\tilde{\nu}) > 0 \qquad \text{if } m^{(k)}_{\tilde{\ell}^{(k-1)}-1} = 0 \text{ and } \dot{\ell}^{(k-1)} = \tilde{\ell}^{(k-1)} = \tilde{\ell}^{(k)} \le \dot{\ell}^{(k)}$$

$$(1.7) \qquad p^{(k)}_{\tilde{s}^{(k)}-1}(\tilde{\nu}) > 0 \qquad \text{if } m^{(k)}_{\tilde{s}^{(k-1)}-1} = 0 \text{ and } \dot{\ell}^{(k-1)} = \tilde{s}^{(k-1)} = \tilde{s}^{(k)} \le \dot{\ell}^{(k)}.$$

Up to minor modifications, the proofs of (1.4)-(1.7) go through as the proofs of [1, (A.2) and (A.3)].

The cases $k = n-1$ and $k = n$ can be proven in a similar fashion.

Case (ℓb)(1). The proof follows very closely the doubly singular case (1) in [1, Appendix A]. Again we assume that $k \le n-2$. The cases $k = n-1, n$ go through up to minor modifications. By assumption $\dot{\tilde{\ell}}^{(k)} < \ell$. By the same arguments as in [1, Appendix A] it follows that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ and $p^{(k)}_{\ell-1}(\dot{\nu}) = 0$.

First we show that the cases I.(ℓs)(1),(1'),(2), II.(1'-3') cannot occur at $k-1$. If II.(1'-3') holds at $k-1$ and the conditions of I.(ℓb)(1) at k, then $\tilde{\ell}^{(k-1)} = \dot{s}^{(k-1)} = \tilde{\ell}^{(k)} = \dot{\ell}^{(k)} = \ell$. For case II.(1') at $k-1$, we have $\dot{\tilde{\ell}}^{(k-1)} = \ell - 1$ so that $p^{(k-1)}_{\ell-1}(\dot{\nu}) = 0$. Otherwise this yields a contradiction to the fact that $\tilde{\ell}^{(k-1)} = \ell$. But $p^{(k-1)}_{\ell-1}(\dot{\nu}) = p^{(k-1)}_{\ell-1} + \chi(\dot{\ell}^{(k-1)} \le \ell - 1 < \dot{\ell}^{(k)}) = p^{(k-1)}_{\ell-1} + 1 \ge 1$. On the other hand for case II.(2'-3') $\dot{\tilde{\ell}}^{(k)} \ge \dot{\tilde{\ell}}^{(k-1)} \ge \tilde{\ell}^{(k-1)} = \ell$ which contradicts our assumptions that $\dot{\tilde{\ell}}^{(k)} < \ell$. Case I.($\ell s$)(2) at $k-1$ requires case I.(ℓs)(2) at k which contradicts our assumption. If I.(ℓs)(1) holds at $k-1$, then $\tilde{\ell}^{(k-1)} = \dot{\ell}^{(k-1)} \le \ell$ and $\dot{\tilde{\ell}}^{(k-1)} \ge \ell$ which contradicts our assumption that $\dot{\tilde{\ell}}^{(k)} < \ell$ since $\dot{\tilde{\ell}}^{(k-1)} \le \dot{\tilde{\ell}}^{(k)}$. Similarly, for I.($\ell s$)(1') $\dot{\tilde{\ell}}^{(k-1)} = \ell$ which contradicts $\dot{\tilde{\ell}}^{(k)} < \ell$.

The goal is to show that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$. Since $\tilde{\ell}^{(k)} = \ell$, it follows that $m^{(k)}_{\ell-1}(\tilde{\nu}) \ge 1$. It suffices to show that $\dot{\tilde{\ell}}^{(k-1)} \le \ell - 1$ and $p^{(k)}_{\ell-1}(\tilde{\nu}) = 0$. By the same arguments as in [1, Appendix A] this implies that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$. Note that, since $p^{(k)}_{\ell-1}(\dot{\nu}) = 0$,

$$(1.8) \qquad p^{(k)}_{\ell-1} = p^{(k)}_{\ell-1}(\tilde{\nu}) + \chi(\tilde{\ell}^{(k-1)} < \ell) = \chi(\dot{\ell}^{(k-1)} < \ell).$$

Suppose that $\dot{\tilde{\ell}}^{(k-1)} \ge \ell$. Now $\dot{\tilde{\ell}}^{(k-1)} \le \dot{\tilde{\ell}}^{(k)} = \ell - 1$ so $\tilde{\tilde{\ell}}^{(k-1)} \ne \dot{\tilde{\ell}}^{(k-1)}$. By induction case I.($\ell$a) or II.(1-3) has to hold at $k-1$ (since we showed before that cases I.(ℓs)(1),(1'),(2) and II.(1'-3') cannot occur). In case I.(ℓa) this yields a contradiction by the same reasoning as in [1, Appendix A]. In case II.(1-3) we have $\dot{\ell}^{(k-1)} = \dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \tilde{s}^{(k-1)} = \ell$ and $\tilde{\tilde{\ell}}^{(k-1)} = \tilde{\ell}^{(k-1)} < \ell$ which yields a contradiction in the evaluation of (1.8). Hence $\dot{\tilde{\ell}}^{(k-1)} < \ell$.

Next suppose that $p_{\ell-1}^{(k)}(\tilde{\nu}) \geq 1$. Then by (1.8), $\tilde{\ell}^{(k-1)} \geq \ell$ and $\dot{\ell}^{(k-1)} \leq \ell - 1$. Since $\tilde{\ell}^{(k-1)} \neq \dot{\ell}^{(k-1)}$, by induction case I.($\ell$a) or II.(1-3) holds at $k-1$. As before, cases II.(1-3) yield a contradiction in evaluating (1.8). For cases I.(ℓa) one obtains a contradiction as in [1, Appendix A]. Hence $\dot{\tilde{\ell}}^{(k-1)} < \ell$ and $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$ which implies $\dot{\tilde{\ell}}^{(k)} = \tilde{\ell}^{(k)} = \ell - 1$.

The proof that $m_{\ell-1}^{(k+1)} = 0$ is the same as in [1, Appendix A].

The case $\dot{\tilde{\ell}}^{(k)} < \ell$ is obtained by the application by θ.

Case (ℓb)(2). By assumption $\tilde{\ell}^{(k)} = \ell$, so that by case I.($\ell$b)(1) $\dot{\tilde{\ell}}^{(k)} \geq \ell$. In addition $m_\ell^{(k)} \geq 2$ and $p_\ell^{(k)} = 0$. By (1.2) $\dot{\tilde{\ell}}^{(k-1)} \leq \ell$ unless case I.(ℓs)(1') holds at $k-1$ and k. Since $m_\ell^{(k)} \geq 2$ and $p_\ell^{(k)} = 0$, we have $\dot{\tilde{\ell}}^{(k)} = \ell$ so that case I.(ℓb)(2) holds, unless $\tilde{s}^{(k)} = \ell$ and $m_\ell^{(k)} = 2$.

Hence let us from now on assume that $\tilde{s}^{(k)} = \ell$ and $m_\ell^{(k)} = 2$. Note that in this case $k \leq n-2$. We will show that case I.(ℓs)(1') holds with $\ell = \ell'$. Note that $\dot{s}^{(k)} > \ell$, since by assumption $\tilde{\ell}^{(k)} = \ell$. Note that $m_\ell^{(k+1)} \geq 2$ since $\tilde{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \ell$, and by (1.1)

$$p_{\ell-1}^{(k)} + p_{\ell+1}^{(k)} + m_\ell^{(k-1)} + (m_\ell^{(k+1)} - 2) \leq 2 \qquad \text{for } k < n-2$$
$$p_{\ell-1}^{(n-2)} + p_{\ell+1}^{(n-2)} + m_\ell^{(n-3)} + (m_\ell^{(n-1)} + m_\ell^{(n)} - 2) \leq 2.$$

By similar arguments as in the proof [1, Appendix A case (3)] of type A it follows that

$$\begin{aligned} &p_{\ell+1}^{(k)} = 0 \\ (1.9) \qquad &p_{\ell-1}^{(k)} = 2 - m_\ell^{(k-1)} \\ &m_\ell^{(k+1)} = 2 \quad \text{for } k < n-2 \text{ or} \qquad m_\ell^{(n-1)} = m_\ell^{(n)} = 1 \quad \text{for } k = n-2. \end{aligned}$$

Let $\ell'' > \ell$ be minimal such that $m_{\ell''}^{(k)} > 0$. If no such ℓ'' exists, set $\ell'' = \infty$. By (1.1) it follows that $p_i^{(k)} = 0$ for $\ell \leq i \leq \ell''$ and $m_i^{(k-1)} = m_i^{(k+1)} = 0$ for $\ell < i < \ell''$. Hence $\dot{\ell}^{(k)} = \ell''$.

First assume that $\dot{\ell}^{(k+1)} > \ell$. We write down the arguments for $k < n-2$. The case $k = n-2$ is analogous. Note that then case I.(ℓa) and I.(sa) holds at $k+1$ so that by induction $\tilde{s}^{(k+1)} \leq \tilde{s}^{(k)} = \ell$. Since on the other hand $\ell = \tilde{\ell}^{(k)} \leq \tilde{s}^{(k+1)}$, it follows that $\tilde{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \ell$. Since $\tilde{s}^{(k+1)} = \ell$ and $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \ell$ and $m_\ell^{(k)} = 2$, it follows that $\tilde{\dot{s}}^{(k)} > \ell$. Since $m_i^{(k)} = 0$ for $\ell < i < \ell''$ and $p_{\ell''}^{(k)} = 0$, we have $\tilde{\dot{s}}^{(k)} = \ell''$ unless $\dot{s}^{(k)} = \ell''$ and $m_{\ell''}^{(k)} = 1$. We deal with this case later. In addition, since $m_i^{(k+1)} = 0$ for $\ell < i < \ell''$, it follows that $\dot{\tilde{\ell}}^{(k)} = \ell'' \leq \dot{\ell}^{(k+1)}$.

If $\dot{\ell}^{(k+1)} = \ell$, then $\tilde{\ell}^{(k+1)} = \ell$. (Note that in this case $k < n-2$, since for $k = n-2$ we have $\dot{\ell}^{(n-1)} = \dot{\ell}^{(n)} = \ell$, which would imply that $\dot{s}^{(n-2)} = \ell$. However this contradicts $\tilde{\ell}^{(n-2)} = \ell$ since $m_\ell^{(n-2)} = 2$). Furthermore by (1.1), $m_i^{(k)} = m_i^{(k+1)} = 0$ for $\ell < i < \ell''$ and $m_{\ell''}^{(k)}, m_{\ell''}^{(k+1)} > 0$. By the same arguments as above $p_i^{(k+1)} = 0$ for $\ell \leq i \leq \ell''$, so that $\dot{\tilde{\ell}}^{(k+1)} = \ell''$. Hence case I.($\ell s$)(1') holds at $k+1$ with the same values for $\ell = \ell'$ and ℓ''. By induction $\tilde{\dot{s}}^{(k+1)} = \ell''$, so that $\tilde{\dot{s}}^{(k)} = \ell''$ as claimed unless again $\dot{s}^{(k)} = \ell''$ and $m_{\ell''}^{(k)} = 1$.

By (1.3) we have $\dot{\tilde{s}}^{(k+1)} \leq \dot{s}^{(k)}$ unless possibly case I.(ℓs)(1') holds at k and $k+1$. However, if case I.(ℓs)(1') holds at $k+1$ by induction $\dot{\tilde{s}}^{(k+1)} = \dot{s}^{(k+1)} \leq \dot{s}^{(k)}$. Hence by the definition of δ also $\dot{\tilde{s}}^{(k)} = \dot{s}^{(k)}$ unless $\dot{s}^{(k)} = \ell''$ and $m_{\ell''}^{(k)} = 1$.

Suppose $\dot{s}^{(k)} = \ell''$ and $m^{(k)}_{\ell''} = 1$. Then $\dot{\tilde{\ell}}^{(k)} = \dot{s}^{(k)} = \ell''$ and $\tilde{\dot{\ell}}^{(k)} > \ell''$. Let $\ell''' > \ell''$ be minimal such that $m^{(k)}_{\ell'''} > 0$. By (1.1) with $p^{(k)}_{\ell''-1} = p^{(k)}_{\ell''} = 0$

$$m^{(k-1)}_{\ell''} + (m^{(k+1)}_{\ell''} - 2) + p^{(k)}_{\ell''+1} \leq 2. \tag{1.10}$$

Note that $m^{(k+1)}_{\ell''} \geq 2$. Assume that $m^{(k+1)}_{\ell''} = 1$ (since $\dot{s}^{(k)} = \ell''$ we must have $m^{(k+1)}_{\ell''} \geq 1$). Then by (1.1) $m^{(a)}_{\ell''} = 1$ for all $k \leq a \leq n-2$. However this is a contradiction to the fact that $\dot{\ell}^{(a)} = \dot{s}^{(a)} = \ell''$ for some $a \geq k$. This proves in particular that case I.(ℓs)(1') cannot hold at $k-1$. Furthermore, by (1.10) $m^{(k-1}_{\ell''} = 0$, $m^{(k+1)}_{\ell''} = 2$ and $p^{(k)}_{\ell''+1} = 0$. Using (1.1) once again this implies $p^{(k)}_i = 0$ for $\ell'' \leq i \leq \ell'''$, so that $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell'''$. Note that $m^{(k-1)}_i = 0$ for $\ell' < i < \ell'''$ in this case.

It remains to show that $\tilde{\dot{s}}^{(k)} \leq \tilde{s}^{(k-1)}$ or case I.(ℓs)(1') holds at $k-1$ with the same values of $\ell = \ell'$ and ℓ''. Since $m^{(k-1)}_i = 0$ for $\ell < i < \ell''$ (resp. for $\ell < i < \ell'''$ in the special case that $\dot{s}^{(k)} = \ell''$ and $m^{(k)}_{\ell''} = 1$), it follows that $\tilde{s}^{(k-1)} \geq \ell'' = \tilde{\dot{s}}^{(k)}$ (resp. $\tilde{s}^{(k-1)} \geq \ell''' = \tilde{\dot{s}}^{(k)}$) if $\tilde{s}^{(k-1)} > \ell$. Hence assume that $\tilde{s}^{(k-1)} = \ell$.

If $\tilde{\ell}^{(k-1)} = \ell$, then $m^{(k-1)}_\ell \geq 2$ and by (1.9) $m^{(k-1)}_\ell = 2$ and $p^{(k)}_{\ell-1} = 0$. Let $v < \ell$ be maximal such that $m^{(k)}_v > 0$. Then by (1.1) $m^{(k-1)}_i = m^{(k+1)}_i = 0$ for $v < i < \ell$ and $p^{(k)}_i = 0$ for $v \leq i \leq \ell$. Hence, if $\dot{\ell}^{(k-1)} < \ell$, then $\dot{\ell}^{(k-1)} \leq v$ and $\dot{\ell}^{(k)} = v < \ell$ since $p^{(k)}_v = 0$ which is a contradiction to our definition $\dot{\ell}^{(k)} = \ell$. Hence $\dot{\ell}^{(k-1)} = \ell$ and case I.(ℓs)(1') holds at $k-1$ with the same value for $\ell = \ell'$. Also ℓ'' is the same by (1.1).

Next assume $\tilde{\ell}^{(k-1)} < \ell$. Then $m^{(k-1)}_\ell \geq 1$ and $0 \leq p^{(k)}_{\ell-1} \leq 1$ by (1.9). Note that $p^{(k)}_{\ell-1}(\dot{\nu}) = p^{(k)}_{\ell-1} - \chi(\dot{\ell}^{(k-1)} < \ell)$. If $\dot{\ell}^{(k-1)} < \ell$, this implies that $p^{(k)}_{\ell-1} = 1$ and $p^{(k)}_{\ell-1}(\dot{\nu}) = 0$. By induction case I. must hold at $k-1$ and $\tilde{\dot{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)} = \ell$. If $\tilde{\dot{\ell}}^{(k-1)} < \ell$ this implies that $\tilde{\dot{\ell}}^{(k)} \leq \ell - 1$ which contradicts our assumption that $\tilde{\dot{\ell}}^{(k)} = \ell$. The condition $\tilde{\dot{\ell}}^{(k-1)} = \ell$ can only occur for case I.(ℓb)(3) at $k-1$. However, then $\dot{\tilde{\ell}}^{(k-1)} = \tilde{\dot{\ell}}^{(k-1)} = \ell$ which contradicts $m^{(k-1)}_\ell = 1$. Hence $\dot{\ell}^{(k-1)} = \ell$. The case $p^{(k)}_{\ell-1} = 0$ yields a contradiction as before. Therefore $p^{(k)}_{\ell-1} = 1$ and $m^{(k-1)}_\ell = 1$ by (1.9), so that case II.(1-3) must hold at $k-1$. Note that $p^{(k)}_{\ell-1}(\tilde{\nu}) = p^{(k)}_{\ell-1} - 1 = 0$. Hence if case II.(1) holds at $k-1$, $\dot{\tilde{\ell}}^{(k-1)} = \ell - 1$ so that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ which contradicts our assumptions. For case II.(2) at $k-1$ we must have $\dot{\tilde{\ell}}^{(k-1)} = \ell$ which however contradicts $m^{(k-1)}_\ell = 1$ and $\tilde{s}^{(k-1)} = \ell$. In case II.(3) we have $\dot{\tilde{\ell}}^{(k-1)} > \dot{\ell}^{(k-1)} = \ell$ which contradicts $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell$.

The case $\dot{\tilde{\ell}}^{(k)} = \ell$ follows from the above by the application of θ.

Case (ℓb)(3). By (1.2) either $\tilde{\dot{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)}$ and $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ or case I.(ℓs) holds at $k-1$ and k. The latter case will be dealt with in the proof of case I.(ℓs), hence we assume that $\tilde{\dot{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)}$ and $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$. We follow the proof for type $A^{(1)}_n$ in [1, Appendix A]. By the same arguments as for type A the assumption $m^{(k)}_\ell > 1$ leads to a contradiction unless $m^{(k)}_\ell = 2$ and $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \dot{s}^{(k)} = \tilde{s}^{(k)} = \ell$. Hence either

$$m^{(k)}_\ell = 1 \quad \text{for } k \leq n \text{ or} \tag{1.11}$$

$$m^{(k)}_\ell = 2 \quad \text{and} \quad \dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \dot{s}^{(k)} = \tilde{s}^{(k)} = \ell \quad \text{for } k \leq n-2. \tag{1.12}$$

If (1.11) holds, up to small modifications the arguments for type A yield:

$$p_{\ell+1}^{(k)} = 0 \qquad \text{for } k \le n \tag{1.13}$$

$$p_{\ell-1}^{(k)} = 2 - m_\ell^{(k-1)} \quad \text{for } k \le n-1, \quad p_{\ell-1}^{(n)} = 2 - m_\ell^{(n-2)} \quad \text{for } k = n \tag{1.14}$$

$$m_\ell^{(k+1)} = 0 \qquad \text{for } k < n-2 \quad m_\ell^{(n-1)} = m_\ell^{(n)} = 0 \quad \text{for } k = n-2. \tag{1.15}$$

If (1.12) holds, then by (1.1) we have

$$p_{\ell-1}^{(k)} + p_{\ell+1}^{(k)} + m_\ell^{(k-1)} + (m_\ell^{(k+1)} - 2) \le 2 \qquad \text{for } k < n-2$$
$$p_{\ell-1}^{(n-2)} + p_{\ell+1}^{(n-2)} + m_\ell^{(n-3)} + (m_\ell^{(n-1)} + m_\ell^{(n)} - 2) \le 2 \qquad \text{for } k = n-2$$

since $p_\ell^{(k)} = 0$. Up to small modifications, the type A proof yields that in this case

$$p_{\ell+1}^{(k)} = 0 \tag{1.16}$$

$$p_{\ell-1}^{(k)} = 2 - m_\ell^{(k-1)} \tag{1.17}$$

$$m_\ell^{(k+1)} = 2 \quad \text{for } k < n-2, \qquad m_\ell^{(n-1)} = m_\ell^{(n)} = 1 \quad \text{for } k = n-2. \tag{1.18}$$

Let ℓ' be minimal such that $\ell' > \ell$ and $m_{\ell'}^{(k)} > 0$. If no such ℓ' exists, set $\ell' = \infty$. By (1.13) (resp. (1.16)) $p_\ell^{(k)} = p_{\ell+1}^{(k)} = 0$ so that as a consequence of (1.1)

$$m_i^{(k)} = 0 \qquad \text{for } \ell < i < \ell' \tag{1.19}$$

$$p_i^{(k)} = 0 \qquad \text{for } \ell \le i \le \ell' \tag{1.20}$$

$$m_i^{(k-1)} = m_i^{(k+1)} = 0 \qquad \text{for } \ell < i < \ell' \tag{1.21}$$

and $m_i^{(n)} = 0$ for $\ell < i < \ell'$ and $k = n-2$.

If $\ell' = \infty$, then $\tilde{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \infty$ and, by (1.1) and (1.15) $m_i^{(k+1)} = 0$ for $i \ge \ell$, also $\dot{\ell}^{(k+1)} = \tilde{\ell}^{(k+1)} = \infty$ so that Case I.(ℓb)(3) holds.

Hence assume $\ell' < \infty$. Assume that (1.11) holds. Since $m_\ell^{(k)} = 1$ and $m_i^{(k)} = 0$ for $\ell < i < \ell'$ certainly $\dot{s}^{(k)} \ge \ell'$ and $\tilde{s}^{(k)} \ge \ell'$. First assume that $\dot{s}^{(k)} > \ell'$ and $\tilde{s}^{(k)} > \ell'$ or $m_{\ell'}^{(k)} > 1$. By the same arguments as in type A it follows that $\tilde{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell' \le \dot{\ell}^{(k+1)}, \tilde{\ell}^{(k+1)}$ so that Case I.(ℓb)(3) holds. Up to small modifications these arguments also go through for $k = n-1, n$ and yield $\tilde{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} \le \dot{s}^{(n-2)}, \tilde{s}^{(n-2)}$.

Next consider the case $\tilde{s}^{(k)} = \ell'$, $\dot{s}^{(k)} > \ell'$ and $m_{\ell'}^{(k)} = 1$. This can only occur for $k \le n-2$. We focus here on $k < n-2$. The case $k = n-2$ is obtained by minor notational changes. By induction we have $\tilde{\tilde{\ell}}^{(k-1)} \le \tilde{\ell}^{(k)} = \ell$. Since $\tilde{\tilde{\ell}}^{(k)} > \ell$, $m_i^{(k)} = 0$ for $\ell < i < \ell'$ and $p_{\ell'}^{(k)} = 0$ it follows that $\tilde{\tilde{\ell}}^{(k)} = \ell'$. Furthermore by (1.15) and (1.21) we also have $\tilde{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \tilde{s}^{(k)} = \ell' = \tilde{\tilde{\ell}}^{(k)}$. This is the second string of equalities in case I.(ℓs)(1'). By (1.1) the conditions $m_{\ell'}^{(k)} = 1$ and $p_{\ell'}^{(k)} = 0$ imply

$$p_{\ell'-1}^{(k)} + p_{\ell'+1}^{(k)} + m_{\ell'}^{(k-1)} + m_{\ell'}^{(k+1)} \le 2. \tag{1.22}$$

But since $\tilde{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \ell'$ we have $m_{\ell'}^{(k+1)} \ge 2$, so that by (1.22) $m_{\ell'}^{(k+1)} = 2$, $m_{\ell'}^{(k-1)} = 0$, $p_{\ell'-1}^{(k)} = p_{\ell'}^{(k)} = p_{\ell'+1}^{(k)} = 0$. Let $\ell'' > \ell'$ be minimal such that $m_{\ell''}^{(k)} > 0$. Then by (1.1) $p_i^{(k)} = 0$ for $\ell' \le i \le \ell''$ and $m_i^{(k)} = m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$. By case

I.(ℓb)(1) we must have $\dot{\tilde{\ell}}^{(k)} \geq \ell$ and since $m_\ell^{(k)} = 1$ actually $\dot{\tilde{\ell}}^{(k)} > \ell$. Hence $\dot{\tilde{\ell}}^{(k)} = \ell''$. The condition $m_i^{(k+1)} = 0$ for $\ell \leq i < \ell'$ implies that $\dot{\ell}^{(k+1)} \geq \ell'$.

Assume that $\dot{\ell}^{(k+1)} > \ell'$. Since $m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$ we obtain $\dot{\ell}^{(k+1)} \geq \ell'' = \dot{\tilde{\ell}}^{(k)}$. By induction case I.($\ell$a) and I.($s$a) holds at $k+1$, so that $\tilde{\tilde{s}}^{(k+1)} = \tilde{s}^{(k+1)} = \ell'$ and $\dot{\tilde{s}}^{(k+1)} = \dot{s}^{(k+1)}$. Since $m_{\ell'}^{(k)} = 1$, $m_i^{(k)} = 0$ for $\ell' < i < \ell''$ and $p_{\ell''}^{(k)} = 0$, it follows that $\tilde{\tilde{s}}^{(k)} = \ell''$ unless $\dot{s}^{(k)} = \ell''$ and $m_{\ell''}^{(k)} = 1$. This case can be dealt with in the same way as in the proof of case I.(ℓb)(2). Also $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$. Since $m_i^{(k-1)} = 0$ for $\ell' \leq i < \ell''$, we have $\tilde{s}^{(k-1)} \geq \ell'' = \tilde{\tilde{s}}^{(k)}$. Hence Case I.($\ell s$)(1') holds.

Otherwise $\dot{\ell}^{(k+1)} = \ell'$. In this case by induction case I.(ℓs)(1') holds at $k+1$ since $\dot{\ell}^{(k+1)} = \tilde{\ell}^{(k+1)} = \tilde{\tilde{\ell}}^{(k+1)} = \tilde{s}^{(k+1)} = \tilde{\ell}^{(k+2)} = \ell'$ and $\ell' < \ell'' = \dot{\tilde{\ell}}^{(k)} \leq \dot{\tilde{\ell}}^{(k+1)}$. By (1.1) $m_i^{(k)} = m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$ and $m_{\ell''}^{(k)}, m_{\ell''}^{(k+1)} > 0$. Hence $\tilde{\tilde{s}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \tilde{\tilde{s}}^{(k+1)} = \dot{\tilde{\ell}}^{(k+1)} = \ell''$ unless again $\dot{s}^{(k)} = \ell''$ and $m_{\ell''}^{(k)} = 1$. Furthermore, by induction $\dot{s}^{(k+1)} = \dot{\tilde{s}}^{(k+1)}$, so that also $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ by the definition of δ. Since $m_i^{(k-1)} = 0$ for $\ell' \leq i < \ell''$, we have $\tilde{s}^{(k-1)} \geq \ell'' = \tilde{\tilde{s}}^{(k)}$. Hence Case I.($\ell s$)(1') holds.

Now let $\dot{s}^{(k)} = \tilde{s}^{(k)} = \ell'$ and $m_{\ell'}^{(k)} = 1$. We will show that case I.(ℓs)(2) holds. By (1.15) and (1.21) we have $\tilde{\ell}^{(k+1)}, \dot{\ell}^{(k+1)} \geq \ell'$. Since on the other hand $\dot{s}^{(k)} = \tilde{s}^{(k)} = \ell'$, we must have $\tilde{\ell}^{(k+1)} = \dot{\ell}^{(k+1)} = \ell'$. This yields the second string of equalities in case I.(ℓs)(2). Let $\ell'' > \ell'$ be minimal such that $m_{\ell''}^{(k)} > 0$. If no such ℓ'' exists set $\ell'' = \infty$. Inequality (1.22) holds again, and since $m_{\ell'}^{(k+1)} \geq 2$ due to the fact that $\tilde{s}^{(k+1)} = \tilde{\ell}^{(k+1)} = \ell'$, it follows that $m_{\ell'}^{(k-1)} = 0$, $m_{\ell'}^{(k+1)} = 2$ and $p_{\ell'}^{(k)} = p_{\ell'+1}^{(k)} = 0$. By the usual arguments $m_i^{(k-1)} = m_i^{(k)} = m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$ and $p_i^{(k)} = 0$ for $\ell' \leq i \leq \ell''$. Since case I.(ℓs)(2) cannot hold at $k-1$ since this would imply $m_{\ell'}^{(k)} \geq 2$, we have $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ and $\tilde{\tilde{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)}$. Since $m_\ell^{(k)} = m_{\ell'}^{(k)} = 1$, $\tilde{\ell}^{(k)} = \dot{\ell}^{(k)} = \ell$, $\tilde{s}^{(k)} = \dot{s}^{(k)} = \ell'$ and $p_{\ell''}^{(k)} = 0$, we must have $\dot{\tilde{\ell}}^{(k)} = \tilde{\tilde{\ell}}^{(k)} = \ell''$. Recall that $m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$. Also $m_{\ell'}^{(k)} = 1$, $m_{\ell'}^{(k+1)} = 2$, so that by (1.1) with $i = \ell'$ and $a = k+1$ we have $(m_{\ell'}^{(k+2)} - 2) + p_{\ell'-1}^{(k+1)} + p_{\ell'+1}^{(k+1)} \leq 1$. Note that $p_{\ell'-1}^{(k+1)}(\tilde{\nu}) = p_{\ell'-1}^{(k+1)} - 1$ which implies that $p_{\ell'-1}^{(k+1)} \geq 1$. Hence together with the previous inequality $m_{\ell'}^{(k+2)} = 2$ and $p_{\ell'+1}^{(k+1)} = 0$. By the usual arguments involving (1.1) it follows that $p_i^{(k+1)} = 0$ for $\ell' \leq i \leq \ell''$. Hence $\dot{\tilde{\ell}}^{(k+1)} = \tilde{\tilde{\ell}}^{(k+1)} = \ell''$ and case I.(ℓs)(2) holds at $k+1$. By induction $\dot{\tilde{s}}^{(k+1)} = \tilde{\tilde{s}}^{(k+1)} = \ell''$, so that $\dot{\tilde{s}}^{(k)} = \tilde{\tilde{s}}^{(k)} = \ell''$ if $m_{\ell''}^{(k)} \geq 2$. If $m_{\ell''}^{(k)} = 1$, then let $\ell''' > \ell''$ be minimal such that $m_{\ell'''}^{(k)} > 0$. Since $m_{\ell''}^{(k)} = 1$ and $m_{\ell''}^{(k+1)} = 2$ it follows by (1.1) that $m_{\ell''}^{(k-1)} = p_{\ell''+1}^{(k)} = 0$. Hence $p_i^{(k)} = 0$ for $\ell'' \leq i \leq \ell'''$ and $m_i^{(k-1)} = 0$ for $\ell'' < i < \ell'''$. This implies that $\dot{\tilde{s}}^{(k)} = \tilde{\tilde{s}}^{(k)} = \ell'''$. Furthermore, since $m_i^{(k-1)} = 0$ for $\ell' \leq i < \ell'''$ it follows that $\tilde{s}^{(k-1)} \geq \ell''' = \tilde{\tilde{s}}^{(k)}$ and $\dot{s}^{(k-1)} \geq \ell''' = \dot{\tilde{s}}^{(k)}$. This concludes the proof that case I.(ℓs)(2) holds.

Finally assume that (1.12) holds. Suppose that case I.(ℓs)(2) does not hold at $k-1$. Then by induction $\tilde{\tilde{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)}$ and $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ and by (1.19) and (1.20) $\tilde{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell'$. If case I.($\ell s$)(2) holds at $k-1$, then $\dot{\tilde{\ell}}^{(k-1)} = \tilde{\tilde{\ell}}^{(k-1)} = \ell'$, so that also $\tilde{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell'$. Note that by the restrictions imposed by (1.1) we also have $\tilde{\tilde{\ell}}^{(k+1)} = \dot{\tilde{\ell}}^{(k+1)} = \ell'$ so that case I.(ℓs)(2) holds at $k+1$. By induction $\tilde{\tilde{s}}^{(k+1)} = \dot{\tilde{s}}^{(k+1)} = \ell'$ which implies $\tilde{\tilde{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell'$ unless $m_{\ell'}^{(k)} = 1$. First assume that $m_{\ell'}^{(k)} \geq 2$. If $\dot{\ell}^{(k-1)}, \tilde{\ell}^{(k-1)} < \ell$, then

$p_{\ell-1}^{(k)} \geq 2$ and by (1.17) $m_\ell^{(k-1)} = 0$, so that $\dot{s}^{(k-1)}, \tilde{s}^{(k-1)} \geq \ell' = \tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)}$ and case I.(ℓs)(2) holds. If $\dot{\ell}^{(k-1)} < \ell$ and $\tilde{\ell}^{(k-1)} = \ell$, then $p_{\ell-1}^{(k)} \geq 1$ and by (1.17) $m_\ell^{(k-1)} \leq 1$. Hence $\tilde{s}^{(k-1)} > \ell$. If $\dot{s}^{(k-1)} > \ell$ then as before $\tilde{s}^{(k-1)}, \dot{s}^{(k-1)} \geq \ell' = \dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)}$ and case I.(ℓs)(2) holds. If $\dot{s}^{(k-1)} = \ell$ then case II.(1'-3') holds at $k-1$. Note that $p_{\ell-1}^{(k)}(\dot{\nu}) = p_{\ell-1}^{(k)} - 1 = 0$, so that we need $\tilde{\dot{\ell}}^{(k-1)} \geq \ell$. Since case II.(3') does not hold at k, we must have $\tilde{\dot{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)} = \ell$ so that case II.(2') holds at $k-1$. However, this means $m_\ell^{(k-1)} \geq 2$ which contradicts $\tilde{s}^{(k-1)} > \ell$ since $p_\ell^{(k-1)} = 0$. The case $\dot{\ell}^{(k-1)} = \ell$ and $\tilde{\ell}^{(k-1)} < \ell$ is similar. Finally let $\dot{\ell}^{(k-1)} = \tilde{\ell}^{(k-1)} = \ell$. Then case I.($\ell s$)(2) holds at $k-1$ or $m_\ell^{(k-1)} = 1$ and $\dot{s}^{(k-1)}, \tilde{s}^{(k-1)} > \ell$. In either case all conditions of case I.(ℓs)(2) hold at k.

If $m_{\ell'}^{(k)} = 1$, then by (1.1) $m_{\ell'}^{(k-1)} + m_{\ell'}^{(k+1)} + p_{\ell'-1}^{(k)} + p_{\ell'+1}^{(k)} \leq 2$. By induction case I.($\ell s$)(2) holds at $k+1$ so that $m_{\ell'}^{(k+1)} \geq 2$. This implies that $m_{\ell'}^{(k-1)} = 0$ and $p_{\ell'+1}^{(k)} = 0$. Let $\ell'' > \ell'$ be minimal such that $m_{\ell''}^{(k)} > 0$. Then $p_i^{(k)} = 0$ for $\ell' \leq i \leq \ell''$ and $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell''$. Furthermore, by the same arguments as before $\dot{s}^{(k-1)}, \tilde{s}^{(k-1)} > \ell$ and since $m_i^{(k-1)} = 0$ for $\ell < i < \ell''$ we have $\dot{s}^{(k-1)}, \tilde{s}^{(k-1)} \geq \ell'' = \dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)}$. Hence case I.($\ell s$)(2) holds at k.

Case (ℓs)(1). In the proof of case I.(ℓb)(2,3) we already showed that case I.(ℓs)(1) can occur at k when I.(ℓs)(1) does not occur at $k-1$. In addition we saw that then either $\tilde{\dot{\ell}}^{(k)} \leq \tilde{\ell}^{(k+1)}$ or case I.(ℓs)(1) holds at $k+1$ with the same values of $\ell' = \ell$ and ℓ''. Hence we are left to show that if case I.(ℓs)(1) holds at $k-1$ and k, then either $\tilde{\dot{\ell}}^{(k)} \leq \tilde{\ell}^{(k+1)}$ or case I.(ℓs)(1) holds at $k+1$ with the same values of $\ell' = \ell$ and ℓ''.

Since case I.(ℓs)(1) holds at $k-1$ and k with the same values of ℓ' and ℓ'', we have by (1.1) $m_i^{(k-1)} = m_i^{(k)} = m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$, $m_{\ell''}^{(k)} > 0$ and $p_i^{(k)} = 0$ for $\ell' \leq i \leq \ell''$. By induction we have $m_{\ell'}^{(k)} = 2$ (see proof of case I.(ℓb)(2,3)). Since $\dot{\ell}^{(k)} = \dot{s}^{(k)} = \ell'$, we must also have $\dot{\ell}^{(k+1)} = \dot{s}^{(k+1)} = \ell'$, so that $m_{\ell'}^{(k+1)} \geq 2$. Since case I.(ℓs)(1) holds at $k-1$ we must have $1 \leq m_{\ell'}^{(k-1)} \leq 2$. The case $m_{\ell'}^{(k-1)} = 1$ can only occur if case I.(ℓs)(1) occurs at $k-1$ for the first time and $\dot{\ell}^{(k-1)} = \tilde{\ell}^{(k-1)} < \ell'$. By the change of vacancy numbers this implies that $p_{\ell'-1}^{(k)} \geq 1$ so that by

$$m_{\ell'}^{(k-1)} - 2m_{\ell'}^{(k)} + m_{\ell'}^{(k+1)} + p_{\ell'-1}^{(k)} - 2p_{\ell'}^{(k)} + p_{\ell'+1}^{(k)} \leq 0$$

$m_{\ell'}^{(k+1)} = 2$. We obtain the same conclusion if $m_{\ell'}^{(k-1)} = 2$. If $\tilde{\ell}^{(k+1)} > \ell'$, then $\tilde{\ell}^{(k+1)} \geq \ell''$ since $m_i^{(k+1)} = 0$ for $\ell' < i < \ell''$. In this case $\tilde{\dot{\ell}}^{(k)} = \ell'' \leq \tilde{\ell}^{(k+1)}$ as claimed. If $\tilde{\ell}^{(k+1)} = \ell'$, then $p_{\ell'}^{(k+1)} = 0$ since $\tilde{\ell}^{(k+1)} = \dot{\ell}^{(k+1)} = \ell'$. By (1.1) with $a = k+1$ and $i = \ell'$ it follows that $m_{\ell'}^{(k+2)} = 2$ and $p_{\ell'+1}^{(k+1)} = 0$. Hence again by (1.1) we have $p_i^{(k+1)} = 0$ for $\ell' \leq i \leq \ell''$ which implies that $\tilde{\dot{\ell}}^{(k+1)} = \ell''$. Note that by similar arguments as before it follows that $m_{\ell''}^{(k+1)} = 2$. By induction either $\dot{\tilde{s}}^{(k+2)} = \ell''$ if case I.(ℓs)(1) holds at $k+2$ or $\dot{\tilde{s}}^{(k+2)} \leq \dot{s}^{(k+1)} = \ell'$. Hence $\dot{\tilde{s}}^{(k+1)} = \ell''$ (even if $\tilde{s}^{(k+1)} = \ell''$ then $\dot{\tilde{s}}^{(k+1)} = \ell''$ since $m_{\ell''}^{(k+1)} = 2$). Similarly $\tilde{s}^{(k+1)} = \tilde{\dot{s}}^{(k+1)}$ as claimed.

Case (ℓs)(1'). This case is analogous to case I.(ℓs)(1).

Case (ℓs)(2). In the proof of case I.(ℓb)(3) we already showed that case I.(ℓs)(2) can occur at k when I.(ℓs)(2) does not occur at $k-1$. In addition we saw that then case I.(ℓs)(2) holds at $k+1$ with $\ell = \ell'$ and $\ell''' = \ell''$. Hence we are left to show that case I.(ℓs)(2) holds at $k+1$ if the same case holds at k with the same values of $\ell = \ell'$ and $\ell'' = \ell'''$.

Let $k \leq n-2$. By induction we will show that $m_\ell^{(a)} = 2$ for $k \leq a \leq n-2$ and $m_\ell^{(n-1)} = m_\ell^{(n)} = 1$, $m_i^{(a)} = 0$ for $\ell < i < \ell''$ and $k \leq a \leq n$, and $p_i^{(a)} = 0$ for $\ell \leq i \leq \ell''$ and $k \leq a \leq n$. By induction hypothesis (see proof of case I.(ℓb)(3)) the statements are true for $a = k$. By (1.1) we have

$$m_\ell^{(k-1)} + (m_\ell^{(k+1)} - 2) + p_{\ell-1}^{(k)} + p_{\ell+1}^{(k)} \leq 2 \qquad \text{for } k < n-2$$
$$m_\ell^{(n-3)} + (m_\ell^{(n-1)} + m_\ell^{(k)} - 2) + p_{\ell-1}^{(n-2)} + p_{\ell+1}^{(n-2)} \leq 2 \qquad \text{for } k = n-2.$$

Since $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \dot{s}^{(k)} = \tilde{s}^{(k)} = \ell$, we must have $m_\ell^{(k+1)} \geq 2$ and $m_\ell^{(n-1)}, m_\ell^{(n)} \geq 1$. If $m_\ell^{(k-1)} \geq 2$, then these inequalities prove that $m_\ell^{(k+1)} = 2$ or $m_\ell^{(n-1)} = m_\ell^{(n)} = 1$. If $m_\ell^{(k-1)} = 1$, then case I.(ℓs)(2) must have occurred at $k-1$ for the first time and $\dot{\ell}^{(k-1)} = \tilde{\ell}^{(k-1)} < \ell$. Hence by the change in vacancy numbers this implies that $p_{\ell-1}^{(k)} \geq 1$, so that again $m_\ell^{(k+1)} = 2$ or $m_\ell^{(n-1)} = m_\ell^{(n)} = 1$. Then by (1.1) with $a = k+1$ and $i = \ell$ it follows that $p_{\ell+1}^{(k+1)} = 0$, so that $p_i^{(k+1)} = 0$ for $\ell \leq i \leq \ell''$. Note that by (1.1) also $m_i^{(k+1)} = 0$ for $\ell < i < \ell''$ and $m_{\ell''}^{(k+1)} > 0$. Hence $\dot{\tilde{\ell}}^{(k+1)} = \tilde{\tilde{\ell}}^{(k+1)} = \ell''$.

Note that $m_{\ell''}^{(k-1)}, m_{\ell''}^{(k)}, m_{\ell''}^{(k+1)} > 0$ since by assumption case I.(ℓs)(2) holds at $k-1$. Assume that $m_{\ell''}^{(k)} = 1$. Then by (1.1)

$$m_{\ell''}^{(k-1)} + m_{\ell''}^{(k+1)} + p_{\ell''+1}^{(k)} \leq 2,$$

which shows that $m_{\ell''}^{(k-1)} = m_{\ell''}^{(k)} = m_{\ell''}^{(k+1)} = 1$. Continuing this by induction one finds by (1.1) with $a = k, k+1, \ldots n-2$ that $m_{\ell''}^{(a)} = 1$ for $k-1 \leq a \leq n-2$ and either $m_{\ell''}^{(n-1)} = 1$ and $m_{\ell''}^{(n)} = 0$ or $m_{\ell''}^{(n-1)} = 0$ and $m_{\ell''}^{(n)} = 1$. Suppose the latter case holds. Then by (1.1) with $a = n-1$ and $i = \ell''$ we have

$$m_{\ell''}^{(n-2)} - 2m_{\ell''}^{(n-1)} + p_{\ell''-1}^{(n-1)} + p_{\ell''+1}^{(n-1)} \leq 0,$$

which yields a contradiction since $m_{\ell''}^{(n-2)} = 1$ and $m_{\ell''}^{(n-1)} = 0$. Hence $m_{\ell''}^{(k)} = 2$ and by induction using (1.1) in fact $m_{\ell''}^{(a)} = 2$ for $k \leq a \leq n-2$, $m_{\ell''}^{(n-1)} = m_{\ell''}^{(n)} = 1$. Hence $\dot{\tilde{\ell}}^{(a)} = \tilde{\tilde{\ell}}^{(a)} = \dot{\tilde{s}}^{(a)} = \tilde{\tilde{s}}^{(a)} = \ell''$ for $k \leq a \leq n-2$ and $\dot{\tilde{\ell}}^{(n-1)} = \dot{\tilde{\ell}}^{(n)} = \tilde{\tilde{\ell}}^{(n-1)} = \tilde{\tilde{\ell}}^{(n)} = \ell''$.

II. Twisted case. Note that this case can only occur for $1 \leq k \leq n-2$. The proof that $\tilde{\tilde{\ell}}^{(k)} = \tilde{\ell}^{(k)}$ goes through as for the generic case of type A in [1, Appendix A].

Case (1). Suppose that $\dot{\tilde{\ell}}^{(k)} < \ell$. By induction $\dot{\tilde{\ell}}^{(k)} \geq \dot{\tilde{\ell}}^{(k-1)} \geq \dot{\ell}^{(k-1)} - 1$. First assume that $\dot{\ell}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} < \ell$. Then δ must select a string shortened by $\tilde{\delta}$ in the transformation $(\nu, J) \to (\tilde{\nu}, \tilde{J})$. By the same arguments as for the generic case in [1, Appendix A], δ does not pick the string of length $\tilde{\ell}^{(k)} - 1$ in $(\tilde{\nu}, \tilde{J})^{(k)}$ shortened by $\tilde{\delta}$.Hence $\dot{\tilde{\ell}}^{(k)} = \ell - 1$. The label of the corresponding string in $(\tilde{\nu}, \tilde{J})^{(k)}$ must be zero since it was shortened by $\tilde{\delta}$ and singular since it is selected by δ. This implies that $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. Next assume that $\dot{\ell}^{(k-1)} - 1 = \dot{\tilde{\ell}}^{(k)} < \ell$. Then case II.(1) or I.(ℓb)(1) must hold at $k-1$, so that by induction hypothesis $m_{\dot{\ell}^{(k-1)}-1}^{(k)} = 0$ or $m_{\dot{\ell}^{(k-1)}-1}^{(k)}(\tilde{\nu}) = 0$. For $\dot{\ell}^{(k-1)} - 1 = \dot{\tilde{\ell}}^{(k)}$ one needs $m_{\dot{\ell}^{(k-1)}-1}^{(k)}(\tilde{\nu}) > 0$, so that $\ell = \dot{\ell}^{(k-1)}$. Hence $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ and $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$ as before.

The goal is to show that $\tilde{\dot{s}}^{(k)} = \ell - 1$. Since $\dot{\ell}^{(k)} = \ell$, it follows that $m^{(k)}_{\ell-1}(\dot{\nu}) \geq 1$. Also $\tilde{\dot{\ell}}^{(k)} = \tilde{\ell}^{(k)}$, so that $m^{(k)}_{\ell-1}(\dot{\nu}) \geq 2$ if $\tilde{\dot{\ell}}^{(k)} = \ell - 1$. Hence it suffices to show that $\tilde{\dot{s}}^{(k+1)} \leq \ell - 1$ and $p^{(k)}_{\ell-1}(\dot{\nu}) = 0$, since then $\tilde{\dot{s}}^{(k)} < \ell$ and by similar arguments as before $\tilde{\dot{s}}^{(k)} = \ell - 1$.

Note that

$$
\begin{aligned}
p^{(k)}_{\ell-1}(\nu) &= p^{(k)}_{\ell-1}(\dot{\nu}) + \chi(\dot{\ell}^{(k-1)} \leq \ell - 1) \\
&= p^{(k)}_{\ell-1}(\tilde{\nu}) + \chi(\tilde{s}^{(k+1)} \leq \ell - 1) - \chi(\tilde{\ell}^{(k)} \leq \ell - 1 < \tilde{\ell}^{(k+1)}).
\end{aligned} \tag{1.23}
$$

Since $p^{(k)}_{\ell-1}(\tilde{\nu}) = 0$, $\tilde{\ell}^{(k)} \leq \tilde{\ell}^{(k+1)} \leq \tilde{s}^{(k+1)} \leq \tilde{s}^{(k)} = \ell$ and by construction $p^{(k)}_{\ell-1}(\nu) \geq 0$, this simplifies to

$$
p^{(k)}_{\ell-1}(\nu) = p^{(k)}_{\ell-1}(\dot{\nu}) + \chi(\dot{\ell}^{(k-1)} \leq \ell - 1) = \chi(\tilde{s}^{(k+1)} \leq \ell - 1). \tag{1.24}
$$

Suppose that $p^{(k)}_{\ell-1}(\dot{\nu}) \geq 1$. Then by (1.24) we must have $\tilde{s}^{(k+1)} \leq \ell - 1$ and $\dot{\ell}^{(k-1)} \geq \ell$. Since $\tilde{\ell}^{(k-1)} \leq \tilde{s}^{(k+1)} \leq \ell - 1$, case I.($\ell$a) or II.(1-3) must hold at $k-1$. If $\dot{\tilde{\ell}}^{(k-1)} = \dot{\ell}^{(k-1)} \geq \ell$, this contradicts $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell - 1$. Hence case II.(1) or (3) must hold at $k-1$ and $\dot{\ell}^{(k-1)} = \tilde{s}^{(k-1)} \geq \ell$, so that $\dot{\ell}^{(k-1)} = \dot{\ell}^{(k)} = \tilde{s}^{(k-1)} = \tilde{s}^{(k)} = \ell$. Since $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell - 1$ case II.(1) must hold at $k-1$. But then by (1.24) with k replaced by $k-1$, it follows that $\dot{\ell}^{(k-2)} = \ell$, so that one of case I.(ℓa) and II.(1-3) holds at $k-2$. Since $\dot{\tilde{\ell}}^{(k-2)} \leq \dot{\tilde{\ell}}^{(k-1)} = \ell - 1$, case II.(1) must hold at $k-2$. Repeating this argument we find that $1 = \dot{\ell}^{(0)} = \dot{\ell}^{(1)} = \cdots = \dot{\ell}^{(k)} = \ell$ which contradicts the condition that $\dot{\tilde{\ell}}^{(k)} = \ell - 1 > 0$. Hence $p^{(k)}_{\ell-1}(\dot{\nu}) = 0$.

Suppose that $\tilde{\dot{s}}^{(k+1)} \geq \ell$ and $\tilde{s}^{(k+1)} < \ell$. Then the doubly singular case I.(sb) or the mixed case I.(ℓs) cannot occur at $k+1$ since $\dot{s}^{(k+1)} \geq \dot{\ell}^{(k)} = \ell$, but $\tilde{s}^{(k+1)} < \ell$. Also the generic case I.(sa) cannot occur since then $\tilde{s}^{(k+1)} = \tilde{\dot{s}}^{(k+1)}$ which contradicts our assumptions. Case II. also cannot occur since $\dot{\ell}^{(k+1)} \geq \ell > \tilde{s}^{(k+1)}$ and $\dot{s}^{(k+1)} \geq \ell > \tilde{\ell}^{(k+1)}$. Hence $\tilde{\dot{s}}^{(k+1)} \geq \ell$ and $\tilde{s}^{(k+1)} < \ell$ is impossible.

Next suppose that $\tilde{\dot{s}}^{(k+1)} \geq \ell$ and $\tilde{s}^{(k+1)} = \ell$. By (1.24) this implies that $\dot{\ell}^{(k-1)} = \ell$. Case I.($\ell$a) cannot hold at $k-1$ since then $\dot{\ell}^{(k-1)} = \dot{\tilde{\ell}}^{(k-1)} = \ell$ which contradicts $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell - 1$. Similarly for cases I.(ℓb)(2-3) and II.(2-3) $\dot{\tilde{\ell}}^{(k-1)} \geq \ell$ which contradicts $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell - 1$. Similarly, for the mixed case I.(ℓs) we have $\dot{\tilde{\ell}}^{(k-1)} \geq \ell = \dot{\ell}^{(k-1)}$ which contradicts our assumptions. If case I.(ℓb)(1) holds at $k-1$, then $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)}$ so that case I.(ℓb)(1) and case II.(1) holds at k which contradicts our assumption. Hence case II.(1) must hold at $k-1$. Since by definition $\tilde{\dot{s}}^{(k)} \geq \tilde{\dot{s}}^{(k+1)} \geq \ell$, the same arguments yield that case II.(1) holds at $k-2$ with $\dot{\ell}^{(k-2)} = \ell$. Repeating this argument we find that $1 = \dot{\ell}^{(0)} = \dot{\ell}^{(1)} = \cdots = \dot{\ell}^{(k)} = \ell$ which contradicts the condition that $\dot{\tilde{\ell}}^{(k)} = \ell - 1 > 0$. Hence $\tilde{\dot{s}}^{(k+1)} < \ell$.

This completes the proof that $\tilde{\dot{s}}^{(k)} = \ell - 1$.

Next we will show that $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$. By induction $\dot{s}^{(k)} \geq \dot{\tilde{s}}^{(k+1)} \geq \dot{s}^{(k+1)} - 1$, so that by the definition of the algorithm for δ also $\dot{s}^{(k)} \geq \dot{\tilde{s}}^{(k)}$. If $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ we are done. First assume that $\dot{s}^{(k)} > \dot{\tilde{s}}^{(k)} \geq \dot{s}^{(k+1)}$. Since by the definition of δ there are no singular strings of length $\dot{s}^{(k)} > i \geq \dot{s}^{(k+1)}$ in $(\nu, J)^{(k)}$, this is only possible if the string shortened by $\tilde{\delta}$ is the one selected by δ to obtain $\dot{\tilde{s}}^{(k)}$. However, this is impossible since by the definitions and assumptions $\dot{s}^{(k+1)} \geq \dot{\ell}^{(k)} = \tilde{s}^{(k)} \geq \tilde{\ell}^{(k)}$. Hence assume that $\dot{s}^{(k)} > \dot{\tilde{s}}^{(k)} = \dot{s}^{(k+1)} - 1$. Then case I.($s$b)(1) or II.(1') must hold at $k+1$. If case I.(sb)(1) holds, then $m_{\ell-1}^{(k)} = 0$ and $\tilde{s}^{(k+1)} = \dot{s}^{(k+1)}$. Since by assumption case II.(1) holds at k, we must have $\tilde{s}^{(k+1)} = \dot{s}^{(k+1)} = \dot{\ell}^{(k+1)} = \ell$. Similarly for case II.(1') we must have $\tilde{s}^{(k+1)} = \dot{s}^{(k+1)} = \dot{\ell}^{(k+1)} = \ell$ and either $m_{\ell-1}^{(k)} = 0$ or $m_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. Since we already showed that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ we must have $m_{\ell-1}^{(k)}(\tilde{\nu}) > 0$. Hence both cases yield $m_{\ell-1}^{(k)} = 0$ which implies that $m_{\ell-1}^{(k)}(\tilde{\nu}) \leq 1$ (note that $\tilde{\ell}^{(k)} < \ell$ since otherwise case I.(ℓb) holds at k). But $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ so that $\dot{s}^{(k)} > \dot{\tilde{s}}^{(k)} = \dot{s}^{(k+1)} - 1 = \ell - 1$ is impossible.

It remains to show that $m_{\ell-1}^{(k+1)} = 0$ or $m_{\ell-1}^{(k+1)}(\tilde{\nu}) = 0$, and $m_{\ell-1}^{(k-1)} = 0$ or $m_{\ell-1}^{(k-1)}(\dot{\nu}) = 0$.

With $p_{\ell-1}^{(k)}(\dot{\nu}) = 0$ equation (1.24) becomes

$$p_{\ell-1}^{(k)}(\nu) = \chi(\dot{\ell}^{(k-1)} \leq \ell - 1) = \chi(\tilde{s}^{(k+1)} \leq \ell - 1). \tag{1.25}$$

First assume that $p_{\ell-1}^{(k)}(\nu) = 0$. Then $\dot{\ell}^{(k-1)} = \tilde{s}^{(k+1)} = \ell$. Since $\tilde{\ell}^{(k-1)} \leq \tilde{\ell}^{(k)} < \ell$ case I.(ℓa) or II. must hold at $k-1$. Since in addition $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\tilde{\ell}}^{(k)} = \ell - 1$, case II.(1) must hold at $k-1$. Certainly $m_{\ell-1}^{(k)}(\tilde{\nu}) > 0$ because $\tilde{s}^{(k)} = \ell$. Hence by induction hypothesis $m_{\ell-1}^{(k)} = 0$, so that by (1.1) $m_{\ell-1}^{(k-1)} = m_{\ell-1}^{(k+1)} = 0$.

Next assume that $p_{\ell-1}^{(k)}(\nu) = 1$. Then by (1.25) $\dot{\ell}^{(k-1)} \leq \ell - 1$ and $\tilde{s}^{(k+1)} \leq \ell - 1$. Since $p_{\ell-1}^{(k)}(\nu) = 1$, there is either a string with label 0 or a singular string of length $\ell - 1$ in $(\nu, J)^{(k)}$ if $m_{\ell-1}^{(k)} > 0$. But then $\dot{\ell}^{(k)} < \ell$ or $\tilde{s}^{(k)} < \ell$ which contradicts our assumptions. Hence $m_{\ell-1}^{(k)} = 0$. By (1.1)

$$p_{\ell-2}^{(k)} + m_{\ell-1}^{(k-1)} + m_{\ell-1}^{(k+1)} \leq 2.$$

If $p_{\ell-2}^{(k)} = 2$, then $m_{\ell-1}^{(k-1)} = m_{\ell-1}^{(k+1)} = 0$ and we are done.

If $p_{\ell-2}^{(k)} = 1$, we have $m_{\ell-1}^{(k-1)} + m_{\ell-1}^{(k+1)} \leq 1$. Let $r < \ell - 1$ be maximal such that $m_r^{(k)} > 0$. If no such r exists, set $r = 0$. Then by (1.1) we have $p_i^{(k)} = 1$ for $r < i < \ell$, $p_r^{(k)} \leq 1$ and $m_i^{(k-1)} = m_i^{(k+1)} = 0$ for $r + 1 < i < \ell - 1$. If $p_r^{(k)} = 1$, then $m_{r+1}^{(k-1)} = m_{r+1}^{(k+1)} = 0$. Suppose that $m_{\ell-1}^{(k-1)} = 0$. Then $\dot{\ell}^{(k-1)} \leq r$. Since by assumption $\dot{\ell}^{(k)} = \ell > r$ the string of length r in $(\nu, J)^{(k)}$ must have label 0. This implies that $\tilde{s}^{(k+1)} > r$ and, since $m_i^{(k+1)} = 0$ for $r < i < \ell - 1$, we have $\tilde{s}^{(k+1)} = \ell - 1$. Since $m_{\ell-1}^{(k-1)} = 0$ implies that $m_{\ell-1}^{(k+1)} = 1$, this shows that $m_{\ell-1}^{(k+1)}(\tilde{\nu}) = 0$. Similarly, if $m_{\ell-1}^{(k+1)} = 0$, then $m_{\ell-1}^{(k-1)}(\dot{\nu}) = 0$. Hence suppose that $p_r^{(k)} = 0$. Then $\dot{\ell}^{(k-1)} > r$ and $\tilde{s}^{(k+1)} > r$ since otherwise $\dot{\ell}^{(k)} \leq r < \ell$ or $\tilde{s}^{(k)} \leq r < \ell$ which contradicts our assumptions. Also by (1.1) $m_{r+1}^{(k-1)} + m_{r+1}^{(k+1)} \leq 1$. Hence either $m_{r+1}^{(k-1)} = 1$, $\dot{\ell}^{(k-1)} = r + 1$, $m_{\ell-1}^{(k+1)} = 1$, $\tilde{s}^{(k+1)} = \ell - 1$ or $m_{r+1}^{(k+1)} = 1$, $\tilde{s}^{(k+1)} = r + 1$, $m_{\ell-1}^{(k-1)} = 1$, $\dot{\ell}^{(k-1)} = \ell - 1$. This implies that either $m_{\ell-1}^{(k-1)} = 0$ and $m_{\ell-1}^{(k+1)}(\tilde{\nu}) = 0$, or $m_{\ell-1}^{(k-1)}(\dot{\nu}) = 0$ and $m_{\ell-1}^{(k+1)} = 0$ as claimed.

Finally assume that $p_{\ell-2}^{(k)} = 0$. If $m_{\ell-2}^{(k)} = 0$, then by (1.1) $-p_{\ell-1}^{(k)} - p_{\ell-3}^{(k)} \geq m_{\ell-2}^{(k-1)} + m_{\ell-2}^{(k+1)}$ which yields a contradiction since $p_{\ell-1}^{(k)} = 1$. Hence $m_{\ell-2}^{(k)} \geq 1$. If $\dot{\ell}^{(k-1)} \leq \ell - 2$ or $\tilde{s}^{(k+1)} \leq \ell - 2$, then $\dot{\ell}^{(k)} \leq \ell - 2$ or $\tilde{s}^{(k)} \leq \ell - 2$ since $p_{\ell-2}^{(k)} = 0$ which contradicts our assumptions. Hence $\dot{\ell}^{(k-1)} = \tilde{s}^{(k+1)} = \ell - 1$. This requires $m_{\ell-1}^{(k-1)} \geq 1$ and

$m_{\ell-1}^{(k+1)} \geq 1$. Since $m_{\ell-1}^{(k-1)} + m_{\ell-1}^{(k+1)} \leq 2$ this implies $m_{\ell-1}^{(k-1)} = 1$ and $m_{\ell-1}^{(k+1)} = 1$, so that $m_{\ell-1}^{(k-1)}(\dot{\nu}) = 0$ and $m_{\ell-1}^{(k+1)}(\tilde{\nu}) = 0$ as claimed.

Case (2). First assume that $\tilde{\dot{s}}^{(k)} \geq \ell$. We will show that then $\tilde{\dot{s}}^{(k)} = \ell$. The assumption $\dot{\tilde{\ell}}^{(k)} = \ell$ implies that $m_\ell^{(k)}(\tilde{\nu}) \geq 1$. Since $\tilde{s}^{(k)} = \ell$, one part of size ℓ is shortened in passing from $\nu^{(k)}$ to $\tilde{\nu}^{(k)}$, so that $m_\ell^{(k)} \geq 2$. Now $p_\ell^{(k)} = 0$, so there is at least one string with label 0 in $\nu^{(k)}$ that is not selected by δ acting on (ν, J). The label of this string remains 0 in passing to $\dot{\nu}^{(k)}$. This shows that there is a string of label 0 and length ℓ in $\dot{\nu}^{(k)}$. Thus to prove $\tilde{\dot{s}}^{(k)} = \ell$, it suffices to show that $\tilde{\dot{s}}^{(k+1)} \leq \ell$. If $\tilde{\dot{s}}^{(k+1)} \leq \tilde{s}^{(k+1)}$ then $\tilde{\dot{s}}^{(k+1)} \leq \tilde{s}^{(k+1)} \leq \tilde{s}^{(k)} = \ell$ as desired. Otherwise $\tilde{\dot{s}}^{(k+1)} > \tilde{s}^{(k+1)}$, so that case I.($s$b)(3), I.($\ell s$), II.(3) or (3') holds at $k+1$. By induction $\tilde{\dot{s}}^{(k+1)} \leq \tilde{s}^{(k)} = \ell$.

Next assume that $\tilde{\dot{s}}^{(k)} < \ell$. We will show that this is impossible. By the same arguments as in the proof of case II.(1) the condition $\tilde{\dot{s}}^{(k)} < \ell$ implies that $\tilde{\dot{s}}^{(k)} = \ell - 1$ and $p_{\ell-1}^{(k)}(\dot{\nu}) = 0$. The goal is to show that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ which contradicts the assumption that $\dot{\tilde{\ell}}^{(k)} = \ell$. Similar to the proof of case II.(1), to prove $\dot{\tilde{\ell}}^{(k)} = \ell - 1$ it suffices to show that $\dot{\tilde{\ell}}^{(k-1)} \leq \ell - 1$ and $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$.

Note that (1.23) becomes

$$
\begin{aligned}
p_{\ell-1}^{(k)}(\nu) &= \chi(\dot{\ell}^{(k-1)} \leq \ell - 1) \\
&= p_{\ell-1}^{(k)}(\tilde{\nu}) + \chi(\tilde{s}^{(k+1)} \leq \ell - 1) - \chi(\tilde{\ell}^{(k)} \leq \ell - 1 < \tilde{\ell}^{(k+1)}).
\end{aligned}
\tag{1.26}
$$

Suppose that $p_{\ell-1}^{(k)}(\tilde{\nu}) \geq 1$. Since the top line can be at most one and $\tilde{\ell}^{(k)} \leq \tilde{\ell}^{(k+1)} \leq \tilde{s}^{(k+1)}$, (1.26) implies that $\tilde{s}^{(k+1)} = \ell$. Note that $\tilde{\dot{s}}^{(k+1)} \leq \tilde{\dot{s}}^{(k)} = \ell - 1$, so that $\tilde{\dot{s}}^{(k+1)} < \tilde{s}^{(k+1)} = \ell$. This implies that case I.(sb)(1) or II.(1) holds at $k+1$. In both cases $\ell = \dot{\ell}^{(k)} = \dot{\ell}^{(k+1)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)}$ and $\tilde{\dot{s}}^{(k+1)} = \ell - 1$. Hence $p_{\ell-1}^{(k+1)}(\dot{\nu}) = 0$ and by (1.26) with k replaced by $k+1$ also $p_{\ell-1}^{(k+1)} = 0$. Suppose that $\tilde{s}^{(k+2)} < \ell$ and let $r < \ell$ be maximal such that $m_r^{(k+1)} > 0$. Then by definition $m_i^{(k+1)} = 0$ for $r < i < \ell$ and by (1.1) $p_i^{(k+1)} = 0$ for $r \leq i \leq \ell$ and $m_i^{(k+2)} = 0$ for $r < i < \ell$. However, since by assumption $\tilde{s}^{(k+2)} < \ell$, this means that $\tilde{s}^{(k+2)} \leq r$. In addition, since $p_r^{(k+1)} = 0$, there is a string with label 0 of length r in $(\nu, J)^{(k+1)}$. Hence $\tilde{s}^{(k+1)} \leq r < \ell$ which is a contradiction to the previously shown fact that $\tilde{s}^{(k+1)} = \ell$. Therefore $\tilde{s}^{(k+2)} = \ell$. Repeating similar arguments one finds that $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)} = \cdots = \dot{\ell}^{(n-1)} = \dot{\ell}^{(n)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \cdots = \tilde{s}^{(n-2)} = \tilde{\ell}^{(n)} = \tilde{\ell}^{(n-1)} = \cdots \tilde{\ell}^{(k)} = \ell$. However this yields a contradiction since then case I.(ℓb) holds at k instead of case II.(2). Hence $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$.

Next we need to show that $\dot{\tilde{\ell}}^{(k-1)} \leq \ell - 1$. Suppose that $\dot{\tilde{\ell}}^{(k-1)} \geq \ell$. Now $\tilde{\dot{\ell}}^{(k-1)} \leq \tilde{\dot{\ell}}^{(k)} \leq \tilde{\dot{s}}^{(k)} = \ell - 1$, so that $\dot{\tilde{\ell}}^{(k-1)} \neq \tilde{\dot{\ell}}^{(k-1)}$. By induction case I.($\ell$a), I.($\ell s$)(1)(1'), II.(1-3) or II.(1'-3') holds at $k-1$. If case I.(ℓs)(1) holds at $k-1$, then $\tilde{s}^{(k-1)} = \tilde{s}^{(k)} = \dot{\ell}^{(k)} = \ell$ and $\tilde{\dot{\ell}}^{(k-1)} > \ell$ which contradicts our assumptions. For case I.(ℓs)(1') $\tilde{\dot{\ell}}^{(k-1)} = \tilde{s}^{(k-1)} = \dot{\ell}^{(k)} = \tilde{s}^{(k)} = \ell$ which again contradicts $\tilde{\dot{\ell}}^{(k-1)} < \ell$. For all other cases we must have $\dot{\ell}^{(k-1)} = \ell$. By (1.26) this implies that $\tilde{s}^{(k+1)} = \ell$. By similar arguments as before $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)} = \cdots = \dot{\ell}^{(n-1)} = \dot{\ell}^{(n)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \cdots = \tilde{s}^{(n-2)} = \tilde{\ell}^{(n)} = \tilde{\ell}^{(n-1)} = \cdots \tilde{\ell}^{(k)} = \ell$, which yields a contradiction since then case I.(ℓb) holds at k instead of case II.(2). Hence $\dot{\tilde{\ell}}^{(k-1)} \leq \ell - 1$ which in turn implies together with $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$ that $\dot{\tilde{\ell}}^{(k)} = \ell - 1$. This contradicts our assumption that $\dot{\tilde{\ell}}^{(k)} = \ell$.

The proof of $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ is very similar to the proof of this statement for case II.(1). The case $\dot{s}^{(k)} > \dot{\tilde{s}}^{(k)} \geq \dot{s}^{(k+1)}$ is the same as for II.(1). For $\dot{s}^{(k)} > \dot{\tilde{s}}^{(k)} = \dot{s}^{(k+1)} - 1$ one obtains as before that $\tilde{s}^{(k+1)} = \dot{s}^{(k+1)} = \dot{\ell}^{(k+1)} = \ell$. However this yields the contradiction $\ell = \dot{\tilde{\ell}}^{(k)} \leq \dot{\tilde{s}}^{(k)} = \dot{s}^{(k+1)} - 1 = \ell - 1$.

Case (3). Assume that $\dot{\tilde{\ell}}^{(k)} > \ell$. First note that $m_\ell^{(k)} \geq 2$ leads to a contradiction. Namely, if Case II.(3) holds at $k-1$, then by induction hypothesis $m_\ell^{(k)} = 1$. Otherwise, $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ by induction hypothesis (1.2). Since $\dot{\ell}^{(k)} = \tilde{s}^{(k)} = \ell$ we must have $p_\ell^{(k)} = 0$. The application of $\tilde{\delta}$ leaves a singular string of length ℓ and label 0 in $\tilde{\nu}^{(k)}$ since $m_\ell^{(k)} \geq 2$. But $\dot{\tilde{\ell}}^{(k-1)} \leq \dot{\ell}^{(k)}$ implies $\dot{\tilde{\ell}}^{(k)} \leq \ell$ which contradicts our assumptions. Hence we must have $m_\ell^{(k)} = 1$ and $p_\ell^{(k)} = 0$. Note in particular that it was shown in the proof of case II.(2), that $\dot{\tilde{s}}^{(k)} < \ell$ implies that $\dot{\tilde{\ell}}^{(k)} < \ell$ which contradicts our assumptions. The case $\dot{\tilde{s}}^{(k)} = \ell$ is not possible due to $m_\ell^{(k)} = 1$. Hence $\dot{\tilde{s}}^{(k)} > \ell$.

With this, inequality (1.1) for $i = \ell$ and $a = k$ reads

$$(1.27) \qquad \begin{aligned} & p_{\ell-1}^{(k)} + p_{\ell+1}^{(k)} + m_\ell^{(k-1)} + m_\ell^{(k+1)} \leq 2 && \text{for } 1 \leq k \leq n-3 \\ & p_{\ell-1}^{(n-2)} + p_{\ell+1}^{(n-2)} + m_\ell^{(n-3)} + m_\ell^{(n-1)} + m_\ell^{(n)} \leq 2 && \text{for } k = n-2. \end{aligned}$$

We will show that

$$(1.28) \qquad p_{\ell+1}^{(k)} = 0 \qquad \text{and} \qquad m_\ell^{(k+1)} = \begin{cases} 1 & \text{if } \tilde{s}^{(k+1)} = \ell \\ 0 & \text{otherwise.} \end{cases}$$

In addition, if $k = n-2$, then the same equation holds for $m_\ell^{(n)}$, and $m_\ell^{(n-1)} = 1$ implies that $m_\ell^{(n)} = 0$ and vice versa.

Let $k < n-2$.

If $m_\ell^{(k-1)} = 2$, then by (1.27) we have $p_{\ell+1}^{(k)} = m_\ell^{(k+1)} = 0$, so we are done.

If $m_\ell^{(k-1)} = 1$ and $p_{\ell-1}^{(k)} = 1$, again by (1.27) we have $p_{\ell+1}^{(k)} = m_\ell^{(k+1)} = 0$. Hence assume $m_\ell^{(k-1)} = 1$ and $p_{\ell-1}^{(k)} = 0$. Note that $p_{\ell-1}^{(k)}(\nu) = p_{\ell-1}^{(k)}(\tilde{\nu}) + \chi(\tilde{s}^{(k+1)} < \ell)$ which implies that $\tilde{s}^{(k+1)} = \ell$ since $p_{\ell-1}^{(k)}(\nu) = 0$ and $p_{\ell-1}^{(k)}(\tilde{\nu}) \geq 0$. But $\tilde{s}^{(k+1)} = \ell$ requires $m_\ell^{(k+1)} \geq 1$ so that by (1.27) again $p_{\ell+1}^{(k)} = 0$ and $m_\ell^{(k+1)} = 1$.

Finally suppose $m_\ell^{(k-1)} = 0$. In this case $\dot{\ell}^{(k-1)} \leq \ell - 1$ and $p_{\ell-1}^{(k)}(\nu) = p_{\ell-1}^{(k)}(\dot{\nu}) + 1$, so that $p_{\ell-1}^{(k)} \geq 1$. If $p_{\ell-1}^{(k)} \geq 2$, then $p_{\ell+1}^{(k)} = m_\ell^{(k+1)} = 0$ by (1.27) as claimed. Hence assume $p_{\ell-1}^{(k)} = 1$. If $\tilde{s}^{(k+1)} = \ell$, then necessarily $m_\ell^{(k+1)} \geq 1$ and by (1.1) $m_\ell^{(k+1)} = 1$ and $p_{\ell+1}^{(k)} = 0$. Now assume that $\tilde{s}^{(k+1)} < \ell$. Recall that $p_{\ell-1}^{(k)}(\nu) = p_{\ell-1}^{(k)}(\tilde{\nu}) + \chi(\tilde{s}^{(k+1)} < \ell)$, which implies that $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$ since $p_{\ell-1}^{(k)}(\nu) = 1$ and $\tilde{s}^{(k+1)} < \ell$. Since $\tilde{s}^{(k)} = \ell$ this implies that there is a singular string of length $\ell - 1$ in $(\tilde{\nu}, \tilde{J})^{(k)}$. Since by assumption $\dot{\tilde{\ell}}^{(k)} > \ell$, we must have $\dot{\tilde{\ell}}^{(k-1)} \geq \ell$, so that $\dot{\tilde{\ell}}^{(k-1)} > \dot{\ell}^{(k-1)}$. Hence by (1.2) case II.(3) must hold at $k-1$. We show that this yields a contradiction. For case II.(3) to hold we must have $\dot{\ell}^{(k-1)} = \tilde{s}^{(k-1)}$. Since $\dot{\ell}^{(k-1)} \leq \dot{\ell}^{(k)} = \ell$ and $\tilde{s}^{(k-1)} \geq \tilde{s}^{(k)} = \ell$ this requires $\dot{\ell}^{(k-1)} = \tilde{s}^{(k-1)} = \ell$. However this contradicts our previous finding that $\dot{\ell}^{(k-1)} < \ell$.

For $k = n-2$ the above arguments go through with minor modifications.

This proves (1.28).

By almost identical arguments it follows that

$$m_\ell^{(k-1)} = \begin{cases} 1 & \text{if } \dot{\ell}^{(k-1)} = \ell \\ 0 & \text{otherwise.} \end{cases} \tag{1.29}$$

Since $p_\ell^{(k)} = p_{\ell+1}^{(k)} = 0$ it follows from (1.1), that if $\ell' > \ell$ and $m_i^{(k)} = 0$ for all $\ell < i < \ell'$, then $p_i^{(k)} = 0$ for $\ell \le i \le \ell'$. Moreover (1.1) implies that $m_i^{(k-1)} = m_i^{(k+1)} = 0$ for $\ell < i < \ell'$.

Suppose that $\nu^{(k)}$ has a string longer than ℓ. Let ℓ' be minimal such that $\ell' > \ell$ and $m_{\ell'}^{(k)} \ge 1$. Note that, since $p_{\ell'}^{(k)} = 0$, the string of length ℓ' in $(\nu, J)^{(k)}$ is singular and has label 0. After the application of $\tilde{\delta}$ this string remains singular with label 0 in $(\tilde{\nu}, \tilde{J})^{(k)}$ since $\ell' > \ell = \tilde{s}^{(k)} \ge \tilde{\ell}^{(k)}$. After the application of δ, a singular string with label 0 of length ℓ' remains in $(\dot{\nu}, \dot{J})^{(k)}$ unless $m_{\ell'}^{(k)} = 1$ and $\dot{s}^{(k)} = \ell'$.

First assume that not both $m_{\ell'}^{(k)} = 1$ and $\dot{s}^{(k)} = \ell'$ hold. We will show that then case II.(3)(i) holds. By induction we have $\dot{\tilde{\ell}}^{(k-1)} \le \ell$ (resp. $\dot{\tilde{s}}^{(k+1)} \le \ell$), unless possibly case II.(3) holds at $k-1$ (resp. case II.(3)(1) $k+1$). If $\dot{\tilde{\ell}}^{(k-1)} > \ell$ and case II.(3) holds at $k-1$, then by induction hypothesis $\dot{\ell}^{(k-1)} = \dot{\ell}^{(k)} = \ell$, $\dot{\tilde{\ell}}^{(k-1)} = \dot{\tilde{\ell}}^{(k)} > \ell$ and $m_\ell^{(k)} = 1$. Note that $m_i^{(k-1)} = m_i^{(k)} = 0$ for $\ell < i < \ell'$, $m_{\ell'}^{(k-1)}, m_{\ell'}^{(k)} > 0$ and $\dot{\tilde{\ell}}^{(k-1)} = \dot{\tilde{\ell}}^{(k)} = \ell'$. Similarly, if $\dot{\tilde{s}}^{(k+1)} > \ell$ and case II.(3)(i) holds at $k+1$, then $\dot{\tilde{s}}^{(k+1)} = \dot{\tilde{\ell}}^{(k+1)} = \dot{\tilde{\ell}}^{(k)} = \ell'$, so that $\dot{\tilde{s}}^{(k)} = \ell'$. Now assume that $\dot{\tilde{\ell}}^{(k-1)} \le \ell$ (resp. $\dot{\tilde{s}}^{(k+1)} \le \ell$). Since by assumption $\dot{\tilde{\ell}}^{(k)} > \ell$ and $\dot{\tilde{s}}^{(k)} > \ell$, it follows that $\dot{\tilde{\ell}}^{(k)} = \ell'$ (resp. $\dot{\tilde{s}}^{(k)} = \ell'$). Moreover, if $\dot{\ell}^{(k+1)} > \ell$, by the previous paragraph $m_i^{(k+1)} = 0$ for $\dot{\ell}^{(k)} = \ell < i < \ell'$, so that $\dot{\ell}^{(k+1)} \ge \ell'$. If $\dot{\ell}^{(k+1)} = \ell$, we must have $m_\ell^{(k+1)} = 1$ so that by (1.28) $\tilde{s}^{(k+1)} = \ell$. Since in addition $\dot{\tilde{\ell}}^{(k+1)} \ge \dot{\tilde{\ell}}^{(k)} > \ell$, case II.(3)(i) holds at $k+1$ with $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)}$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k+1)}$ and $m_\ell^{(k+1)} = 1$. Similarly, if $\tilde{s}^{(k-1)} > \ell$, by the previous paragraph $m_i^{(k-1)} = 0$ for $\ell < i < \ell'$, so that $\tilde{s}^{(k-1)} \ge \ell'$. If $\tilde{s}^{(k-1)} = \ell$, we must have $m_\ell^{(k-1)} = 1$ so that by (1.29) $\dot{\ell}^{(k-1)} = \ell$. Since in addition $\dot{\tilde{s}}^{(k-1)} > \ell$, case II.(3) must hold at $k-1$.

Next assume that $m_{\ell'}^{(k)} = 1$ and $\dot{s}^{(k)} = \ell'$. We will show that then case II.(3)(ii) holds. By (1.1) with $a = k$ and $i = \ell'$, using that $p_{\ell'-1}^{(k)} = p_{\ell'}^{(k)} = 0$, we have

$$\begin{aligned} &p_{\ell'+1}^{(k)} + m_{\ell'}^{(k-1)} + m_{\ell'}^{(k+1)} \le 2 && \text{for } 1 \le k \le n-3 \\ &p_{\ell'+1}^{(n-2)} + m_{\ell'}^{(n-3)} + m_{\ell'}^{(n-1)} + m_{\ell'}^{(n)} \le 2 && \text{for } k = n-2. \end{aligned} \tag{1.30}$$

Note that for $k \le n-3$, since $0 \le m_\ell^{(k+1)} \le 1$ and $m_i^{(k+1)} = 0$ for $\ell < i < \ell'$, we must have $\dot{s}^{(k+1)} = \ell'$, which in turn implies that $m_{\ell'}^{(k+1)} \ge 1$. Similarly for $k = n-2$, it follows that $\max\{\dot{\ell}^{(n-1)}, \dot{\ell}^{(n)}\} = \ell'$ so that $m_{\ell'}^{(n-1)} \ge 1$ or $m_{\ell'}^{(n)} \ge 1$. Hence by (1.30) $0 \le m_{\ell'}^{(k-1)} \le 1$. We distinguish the two cases.

We will show that $m_{\ell'}^{(k-1)} = 1$ leads to a contradiction. By (1.30) the assumption $m_{\ell'}^{(k-1)} = 1$ implies that $m_{\ell'}^{(k+1)} = 1$ for $k \le n-3$ and $m_{\ell'}^{(n-1)} = 1$ or $m_{\ell'}^{(n)} = 1$ for $k = n-2$. Since $\dot{s}^{(k+1)} = \ell'$ and $m_i^{(k+1)} = 0$ for $\ell < i < \ell'$, we must have $\dot{\ell}^{(k+1)} = \ell$ which by (1.28) implies $\tilde{s}^{(k+1)} = \ell$ so that case II.(3) holds at $k+1$. Repeating the argument we must have $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)} = \cdots = \dot{\ell}^{(n-2)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \cdots = \tilde{s}^{(n-2)} = \ell$, $\dot{s}^{(k)} = \dot{s}^{(k+1)} = \cdots = \dot{s}^{(n-2)} = \ell'$, $m_\ell^{(k)} = m_\ell^{(k+1)} = \cdots = m_\ell^{(n-2)} = m_{\ell'}^{(k)} =$

$m_{\ell'}^{(k+1)} = \cdots = m_{\ell'}^{(n-2)} = 1$. By (1.30) and (1.27) for $k = n-2$ and the constraints on $\dot{\ell}^{(n-1)}$ and $\dot{\ell}^{(n)}$, we have $m_\ell^{(n-1)} = m_{\ell'}^{(n)} = 1$, $m_\ell^{(n)} = m_{\ell'}^{(n-1)} = 0$, $\dot{\ell}^{(n-1)} = \ell$ and $\dot{\ell}^{(n)} = \ell'$ or the same with $n-1$ and n interchanged. For concreteness let us assume that the first conditions hold. By (1.1) with $a = n-1$ and $i = \ell'$ we have

$$p_{\ell'-1}^{(n-1)} - 2p_{\ell'}^{(n-1)} + p_{\ell'+1}^{(n-1)} + m_{\ell'}^{(n-2)} - 2m_{\ell'}^{(n-1)} \leq 0. \tag{1.31}$$

Since $m_\ell^{(n-1)} = 1$ it follows from (1.28) that $\dot{\ell}^{(n-1)} = \tilde{\ell}^{(n-1)} = \ell$, so that $p_\ell^{(n-1)} = 0$. By similar arguments as before it follows that $p_i^{(n-1)} = 0$ for $\ell \leq i \leq \ell'$. But this with $m_{\ell'}^{(n-1)} = 0$ and $m_{\ell'}^{(n-2)} = 1$ yields a contradiction to (1.31).

Hence $m_{\ell'}^{(k-1)} = 0$. If $m_{\ell'}^{(k+1)} = 1$ we get a contradiction as in the previous case. Hence by (1.30) $m_{\ell'}^{(k+1)} = 2$ and $p_{\ell'+1}^{(k)} = 0$. By induction we have $\dot{\tilde{\ell}}^{(k-1)} \leq \ell$, unless possibly case II.(3) holds at $k-1$. If case II.(3) holds at $k-1$, then by induction hypothesis $\dot{\ell}^{(k-1)} = \dot{\ell}^{(k)} = \ell$. But $m_i^{(k-1)} = 0$ for $\ell < i \leq \ell'$ which would imply that $m_{\ell'}^{(k)} = 0$ which contradicts our assumptions. Hence $\dot{\tilde{\ell}}^{(k-1)} \leq \ell$ and, since by assumption $\dot{\tilde{\ell}}^{(k)} > \ell$ we must have $\dot{\tilde{\ell}}^{(k)} = \ell' = \dot{s}^{(k)}$ as claimed in case II.(3)(ii). Let $\ell'' > \ell'$ be minimal such that $m_{\ell''}^{(k)} \geq 1$. If no such ℓ'' exists, set $\ell'' = \infty$. Again by (1.1) we have $p_i^{(k)} = 0$ for $\ell' \leq i \leq \ell''$. At $k+1$, either case II.(3)(i) holds with $\dot{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \ell$ and $\dot{\tilde{\ell}}^{(k+1)} = \dot{s}^{(k+1)} = \dot{\tilde{s}}^{(k+1)} = \tilde{\tilde{s}}^{(k+1)} = \ell'$, or $\dot{\ell}^{(k+1)} = \ell'$ and the nontwisted generic case holds. In both cases $\dot{\tilde{s}}^{(k+1)} = \ell'$ so that $\dot{\tilde{s}}^{(k)} = \ell''$. If case II.(3)(i) holds at $k+1$, then $\tilde{\tilde{s}}^{(k+1)} = \ell'$, so that $\tilde{\tilde{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell''$ as claimed for case II.(3)(ii). Otherwise the untwisted generic case holds at $k+1$, so that $\tilde{\tilde{s}}^{(k+1)} = \tilde{s}^{(k+1)} \leq \ell$. We already showed in the proof of case II.(2) that $\tilde{\tilde{s}}^{(k)} < \ell$ implies that $\dot{\tilde{\ell}}^{(k)} < \ell$ which contradicts our assumptions. Hence, since the strings of length ℓ and ℓ' are already selected, $\tilde{\tilde{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell''$. Finally, since by assumption and (1.1) $m_i^{(k-1)} = 0$ for $\ell' \leq i < \ell''$, we have $\dot{s}^{(k-1)} \geq \ell''$. Hence case II.(3)(ii) holds.

Otherwise there is no string in $\nu^{(k)}$ longer than ℓ so that $m_i^{(k)} = 0$ for $i > \ell$. Then $\dot{\tilde{\ell}}^{(k)} = \tilde{\tilde{s}}^{(k)} = \infty$. Moreover, $m_i^{(k-1)} = m_i^{(k+1)} = 0$ for $i > \ell$. Hence if $\dot{\ell}^{(k+1)} > \ell$ (resp. $\tilde{s}^{(k-1)} > \ell$), we must have $\dot{\ell}^{(k+1)} = \infty$ (resp. $\tilde{s}^{(k-1)} = \infty$). If $\dot{\ell}^{(k+1)} = \ell$ (resp. $\tilde{s}^{(k-1)} = \ell$), then $m_\ell^{(k+1)} = 1$ and $\tilde{s}^{(k+1)} = \ell$ by (1.28) (resp. $m_\ell^{(k-1)} = 1$ and $\dot{\ell}^{(k-1)} = \ell$ by (1.29)), so that again Case II.(3) holds at $k+1$ (resp. $k-1$).

Case (1'-3'). These cases follow from II.(1-3) by the application of θ. □

Proof of Lemma 1.2. By Lemma 1.1 we have $\dot{\tilde{\nu}} = \tilde{\nu}$, whose proof will be used repeatedly. We also rely on [1, Lemma A.3].

Selected strings. Consider a string in $(\nu, J)^{(k)}$ that is either selected by δ or $\tilde{\delta}$, or is such that its image under δ (resp. $\tilde{\delta}$) is selected by $\tilde{\delta}$ (resp. δ). It is shown that the image of any such string under both $\tilde{\delta} \circ \delta$ and $\delta \circ \tilde{\delta}$, has the same label. The proof of these statements for cases I.(ℓa), I.(ℓb), I.(sa) and I.(sb) is the same as for the analogous cases in [1, Lemma A.3].

Selected strings, case I.(ℓs)(1). We need to distinguish the case whether case I.(ℓs)(1) occurs for the first time at k or not. First assume that case I.(ℓs)(1) does not occur at $k-1$.

The string $(\ell, 0)$ maps to a string of length $\ell - 1$, with label zero under $\delta \circ \tilde{\delta}$ and singular label under $\tilde{\delta} \circ \delta$. Hence we need to show that $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. By the change in vacancy numbers we have

$$p_{\ell-1}^{(k)}(\tilde{\nu}) = p_{\ell-1}^{(k)} - \chi(\dot{\tilde{\ell}}^{(k-1)} \leq \ell - 1) - \chi(\dot{\ell}^{(k-1)} \leq \ell - 1) \tag{1.32}$$

By (1.9), (1.14) and (1.17), $p_{\ell-1}^{(k)} = 2 - m_\ell^{(k-1)}$. Hence if $m_\ell^{(k-1)} = 2$ and the nonnegativity of vacancy numbers, it follows that $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. If $m_\ell^{(k-1)} = 1$, it follows that $p_{\ell-1}^{(k)} = 1$. We need to show that either $\dot{\ell}^{(k-1)} < \ell$ or $\dot{\tilde{\ell}}^{(k-1)} < \ell$. Since by assumption case I.(ℓs)(1) does not hold at $k-1$, we have $\dot{\tilde{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)} = \ell$ by (1.2) which proves the assertion. Finally, if $m_\ell^{(k)} = 0$, we must have $\dot{\ell}^{(k-1)}, \tilde{\ell}^{(k-1)} < \ell$. Furthermore, $p_{\ell-1}^{(k)} = 2$ and by the same arguments as before $\dot{\tilde{\ell}}^{(k-1)} < \ell$. Hence by (1.32), $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$.

The string $(\ell', 0)$ is mapped to a singular string of length $\ell' - 1$ under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$.

If $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell''$, the string $(\ell'', 0)$ is mapped to a string of length $\ell'' - 1$ of label zero under $\tilde{\delta} \circ \delta$ and of singular label under $\delta \circ \tilde{\delta}$. Hence we need to show that $p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = 0$. Note that

$$p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = p_{\ell''-1}^{(k)} - \chi(\dot{\tilde{s}}^{(k+1)} < \ell'') + \chi(\ell'' \leq \tilde{\ell}^{(k+1)}).$$

By the proof of Lemma 1.1 $p_{\ell''-1}^{(k)} = 0$. If case I.(ℓs)(1) holds at $k+1$, both other terms are zero. If case I.(ℓs)(1) holds at $k+1$, the other two expressions yield -1 and 1 respectively, which proves the assertion. The string $(\tilde{s}^{(k)}, 0)$ is mapped to a string of length $\tilde{s}^{(k)} - 1$ of label 0 under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$.

If $\dot{\tilde{\ell}}^{(k)} = \tilde{s}^{(k)} = \ell''$, the string $(\ell'', 0)$ is mapped to a string of length $\ell'' - 1$ of label 0 under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$. The string $(\ell''', 0)$ is mapped to a string of length $\ell''' - 1$ of label 0 under $\tilde{\delta} \circ \delta$ and singular label under $\delta \circ \tilde{\delta}$. Hence it needs to be shown that $p_{\ell'''-1}^{(k)}(\dot{\tilde{\nu}}) = 0$. By the change in vacancy numbers

$$p_{\ell'''-1}^{(k)}(\dot{\tilde{\nu}}) = p_{\ell'''-1}^{(k)} - \chi(\dot{\tilde{\ell}}^{(k+1)} < \ell''') + \chi(\tilde{s}^{(k-1)} \geq \ell''').$$

By the proof of Lemma 1.1 $p_{\ell'''-1}^{(k)} = 0$ and the value of the other two terms is -1 and 1, respectively, which proves the assertion.

Now suppose that case I.(ℓs)(1) holds at $k-1$. Then by the proof of Lemma 1.1 $m_\ell^{(k)} = m_\ell^{(k+1)} = 2$, $1 \leq m_\ell^{(k-1)} \leq 2$ and $p_\ell^{(k)} = p_{\ell+1}^{(k)} = 0$. Hence by (1.1) $m_\ell^{(k-1)} - 2 + p_{\ell-1}^{(k)} \leq 0$. If $m_\ell^{(k-1)} = 2$, then $p_{\ell-1}^{(k)} = 0$ and by (1.32) also $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. If $m_\ell^{(k-1)} = 1$ we must have $\dot{\ell}^{(k-1)} < \ell$ and by the change in vacancy numbers $p_{\ell-1}^{(k)} \geq 1$. Hence by the previous inequality $p_{\ell-1}^{(k)} = 1$ and by (1.32) $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$. The same is true for the selected string $(\ell', 0)$ since $\ell = \ell'$ in this case. The proof for the selected strings $(\ell'', 0)$ and $(\tilde{s}^{(k)}, 0)$ goes through as before.

Selected strings, case I.(ℓs)(1'). This case is analogous to the proof of case I.(ℓs)(1).

Selected strings, case I.(ℓs)(2). The proof for the string $(\ell, 0)$ is almost identical to the proof for case I.(ℓs)(1). When $\ell' = \ell$, the string $(\ell', 0)$ also changes as required. If $\ell' > \ell$, it needs to be shown that $p_{\ell'-1}^{(k)}(\tilde{\nu}) = 0$. By the change in vacancy number

$$p_{\ell'-1}^{(k)}(\tilde{\nu}) = p_{\ell'-1}^{(k)} + \chi(\dot{\ell}^{(k)} < \ell') - \chi(\dot{\tilde{\ell}}^{(k-1)} < \ell') = 0 + 1 - 1 = 0$$

where we used that $\dot{\tilde{\ell}}^{(k-1)} \leq \tilde{\ell}^{(k)} = \ell$ since for $\ell < \ell'$ case I.(ℓs)(2) does not hold at $k-1$.

The string $(\ell'', 0)$ is mapped to a string of length $\ell'' - 1$ with singular label by $\delta \circ \tilde{\delta}$ and label zero by $\tilde{\delta} \circ \delta$. Hence it needs to be shown that $p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = 0$. The vacancy number changes as

$$p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = p_{\ell''-1}^{(k)} - \chi(\dot{\tilde{\ell}}^{(k-1)} < \ell'') + \chi(\tilde{s}^{(k-1)} \geq \ell'').$$

By the proof of Lemma 1.1 $p^{(k)}_{\ell''-1} = 0$. Except for the first occurrence of case I.(ℓs)(2) the other two terms are zero as well. If case I.(ℓs)(2) occurs at k for the first time, $\dot{\tilde{\ell}}^{(k-1)} \le \dot{\ell}^{(k)} = \ell < \ell''$ and $\tilde{s}^{(k-1)} \ge \ell''$, so that again $p^{(k)}_{\ell''-1}(\dot{\tilde{\nu}}) = 0$ as claimed.

Finally, if $\ell''' > \ell''$ we need to show that $p^{(k)}_{\ell'''-1}(\dot{\tilde{\nu}}) = 0$. The vacancy numbers change as $p^{(k)}_{\ell'''-1}(\dot{\tilde{\nu}}) = p^{(k)}_{\ell'''-1} - \chi(\dot{\tilde{\ell}}^{(k+1)} < \ell''') + \chi(\tilde{\ell}^{(k-1)} \ge \ell''') = 0 - 1 + 1 = 0$ by the details of the proof of Lemma 1.1.

Selected strings, case II.(1). The string $(\tilde{\ell}^{(k)}, 0)$ is mapped to a string of length $\tilde{\ell}^{(k)} - 1$ under both $\tilde{\delta} \circ \delta$ and $\delta \circ \tilde{\delta}$. The singular string of length $\dot{s}^{(k)}$ is mapped to a singular string of length $\dot{s}^{(k)} - 1$ under both $\tilde{\delta} \circ \delta$ and $\delta \circ \tilde{\delta}$.

Finally, the string $(\ell, 0)$ is mapped to a singular string of length $\ell - 2$ under $\delta \circ \tilde{\delta}$ and a string of label 0 of length $\ell - 2$ under $\tilde{\delta} \circ \delta$. Hence we need to show that $p^{(k)}_{\ell-2}(\dot{\tilde{\nu}}) = 0$. By the change in vacancy number $p^{(k)}_{\ell-2}(\dot{\tilde{\nu}}) = p^{(k)}_{\ell-2} - \chi(\tilde{s}^{(k+1)} \le \ell - 2) - \chi(\dot{\tilde{\ell}}^{(k-1)} \le \ell - 2)$.

If $p^{(k)}_{\ell-1} = 0$, then $m^{(k)}_{\ell-1} = 0$ and hence by (1.1) $p^{(k)}_{\ell-2} = 0$. Otherwise by (1.24) $p^{(k)}_{\ell-1} = 1$, $\tilde{s}^{(k+1)} < \ell$ and $\dot{\ell}^{(k-1)} < \ell$. In this case $m^{(k)}_{\ell-1} = 0$ since else $\tilde{\delta}$ or δ would pick a string of length $\ell - 1$ in $(\nu, J)^{(k)}$. Hence by (1.1)

$$m^{(k-1)}_{\ell-1} + m^{(k+1)}_{\ell-1} + p^{(k)}_{\ell-2} + p^{(k)}_{\ell} \le 2. \tag{1.33}$$

If $p^{(k)}_{\ell-2} = 0$, then also $p^{(k)}_{\ell-2}(\dot{\tilde{\nu}}) = 0$ and we are done. Assume that $p^{(k)}_{\ell-2} = 1$. We need to show that either $\tilde{s}^{(k+1)} \le \ell-2$ or $\dot{\ell}^{(k-1)} \le \ell-2$, so that $p^{(k)}_{\ell-2}(\dot{\tilde{\nu}}) = 0$ as required. Suppose that $\tilde{s}^{(k+1)} > \ell - 2$ and $\dot{\ell}^{(k-1)} > \ell - 2$, which implies that $\tilde{s}^{(k+1)} = \dot{\ell}^{(k-1)} = \ell - 1$. But by (1.33) either $m^{(k-1)}_{\ell-1} = 0$ or $m^{(k+1)}_{\ell-1} = 0$ which yields a contradiction. Next assume that $p^{(k)}_{\ell-2} = 2$. In this case (1.33) implies that $m^{(k-1)}_{\ell-1} = m^{(k+1)}_{\ell-1} = 0$, so that $\tilde{s}^{(k+1)}, \dot{\ell}^{(k-1)} \le \ell-2$. Hence $p^{(k)}_{\ell-2}(\dot{\tilde{\nu}}) = p^{(k)}_{\ell-2} - \chi(\tilde{s}^{(k+1)} \le \ell-2) - \chi(\dot{\tilde{\ell}}^{(k-1)} \le \ell-2) = 0$ as required.

Selected strings, case II.(2). The selected string $(\tilde{\ell}^{(k)}, 0)$ is mapped to a string of length $\tilde{\ell}^{(k)} - 1$ with label 0 under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$. The selected singular string of length $\dot{s}^{(k)}$ is mapped to a singular string of length $\dot{s}^{(k)} - 1$ under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$. The selected singular string of length $\dot{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ is mapped to a singular string of length $\dot{\ell}^{(k)} - 1$, and the selected string of length $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$ with label 0 is mapped to a string of length $\tilde{s}^{(k)} - 1$ with label 0 under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$.

Selected strings, case II.(3)(i). The argument for the selected strings of length $\tilde{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ and $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ is the same as in the previous cases. To show that the selected strings of length $\ell = \dot{\ell}^{(k)} = \tilde{s}^{(k)}$ obtain the same label under $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$, it suffices to show that $p^{(k)}_{\ell-1}(\tilde{\dot{\nu}}) = 0$. By the change in vacancy numbers

$$p^{(k)}_{\ell-1}(\tilde{\dot{\nu}}) = p^{(k)}_{\ell-1} - \chi(\dot{\ell}^{(k-1)} \le \ell - 1) - \chi(\tilde{\dot{s}}^{(k+1)} \le \ell - 1) \tag{1.34}$$

and by (1.1)

$$p^{(k)}_{\ell-1} + p^{(k)}_{\ell+1} + m^{(k-1)}_{\ell} + m^{(k+1)}_{\ell} \le 2. \tag{1.35}$$

Hence $p^{(k)}_{\ell-1} \le 2$. If $p^{(k)}_{\ell-1} = 0$, then by (1.34) and the nonnegativity of the vacancy numbers also $p^{(k)}_{\ell-1}(\tilde{\dot{\nu}}) = 0$. If $p^{(k)}_{\ell-1} = 1$, by (1.34) $\dot{\ell}^{(k-1)} = \ell$ or $\tilde{s}^{(k+1)} = \ell$ which requires $m^{(k-1)}_{\ell} = 1$ or $m^{(k+1)}_{\ell} = 1$. By (1.35) this implies that $m^{(k+1)}_{\ell} = 0$ or $m^{(k-1)}_{\ell} = 0$ so that $\tilde{s}^{(k+1)} \le \ell - 1$ or $\dot{\ell}^{(k-1)} \le \ell - 1$. By (1.34) in turn we have $p^{(k)}_{\ell-1}(\tilde{\dot{\nu}}) = 0$. If $p^{(k)}_{\ell-1} = 2$, we must have $m_\ell^{(k-1)} = m_\ell^{(k+1)} = 0$ by (1.35). Hence $\tilde{s}^{(k+1)}, \dot{\ell}^{(k-1)} \le \ell - 1$ and by (1.35) $p_{\ell-1}^{(k)}(\tilde{\nu}) = 0$.

Let $\ell' = \dot{\tilde{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)}$. To show that the selected strings of length ℓ' obtain the same label under $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$, it suffices to show that $p_{\ell'-1}^{(k)}(\dot{\tilde{\nu}}) = 0$. Since $p_{\ell'-1}^{(k)} = 0$, we have $p_{\ell'-1}^{(k)}(\dot{\tilde{\nu}}) = \chi(\tilde{s}^{(k-1)} \ge \ell') - \chi(\dot{\tilde{\ell}}^{(k-1)} < \ell')$. Two cases can hold. Either $\tilde{s}^{(k-1)} \ge \dot{\tilde{s}}^{(k)} = \ell'$ and case II.(3)(i) does not hold at $k-1$ so that $\dot{\tilde{\ell}}^{(k-1)} \le \dot{\ell}^{(k)} = \ell$. In this case $p_{\ell'-1}^{(k)}(\dot{\tilde{\nu}}) = \chi(\tilde{s}^{(k-1)} \ge \ell') - \chi(\dot{\tilde{\ell}}^{(k-1)} < \ell') = 1 - 1 = 0$ as required. Otherwise case II.(3)(i) holds at $k-1$ so that $\dot{\tilde{s}}^{(k-1)} = \dot{\tilde{\ell}}^{(k-1)} = \ell'$ and $\tilde{s}^{(k-1)} = \ell < \ell'$, so that $p_{\ell'-1}^{(k)}(\dot{\tilde{\nu}}) = \chi(\tilde{s}^{(k-1)} \ge \ell') - \chi(\dot{\tilde{\ell}}^{(k-1)} < \ell') = 0 - 0 = 0$.

Selected strings, case II.(3)(ii). The proof for the selected strings of length $\tilde{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ and $\dot{\ell}^{(k)} = \tilde{s}^{(k)} = \ell$ is the same as for case II.(3)(i). The selected string of length $\dot{s}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell'$ is mapped to a singular string of length $\ell' - 1$ under both $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$. To show that the selected string of length $\dot{\tilde{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell''$ obtains the same label under $\delta \circ \tilde{\delta}$ and $\tilde{\delta} \circ \delta$ it needs to be shown that $p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = 0$. Since $p_{\ell''-1}^{(k)} = 0$, we have $p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = \chi(\tilde{s}^{(k-1)} \ge \ell'') - \chi(\dot{\tilde{s}}^{(k+1)} < \ell'')$. Since case II.(3) cannot occur before case II.(3)(ii), it follows from (1.3) that $\tilde{s}^{(k-1)} \ge \dot{\tilde{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell''$. By induction either case II.(3)(i) holds at $k+1$ in which case $\dot{\tilde{s}}^{(k+1)} = \dot{s}^{(k+1)} = \ell' < \ell''$ or $\dot{\tilde{s}}^{(k+1)} \le \dot{s}^{(k)} = \ell < \ell''$. Hence $p_{\ell''-1}^{(k)}(\dot{\tilde{\nu}}) = \chi(\tilde{s}^{(k-1)} \ge \ell'') - \chi(\dot{\tilde{s}}^{(k+1)} < \ell'') = 1 - 1 = 0$ as required.

Unselected strings. For the rest of the proof, assume that (i, x) is a string in $(\nu, J)^{(k)}$ that is not selected by δ or $\tilde{\delta}$, and is such that its image under $\tilde{\delta}$ (resp. δ) is not selected by δ (resp. $\tilde{\delta}$).

Using the fact that δ preserves labels and $\tilde{\delta}$ preserves colabels, it is enough to show that

$$p_i^{(k)}(\nu) - p_i^{(k)}(\tilde{\nu}) = p_i^{(k)}(\dot{\nu}) - p_i^{(k)}(\dot{\tilde{\nu}}), \tag{1.36}$$

which by the change in vacancy numbers is equivalent to

$$\begin{aligned}
&\chi(\tilde{\ell}^{(k-1)} \le i < \tilde{\ell}^{(k)}) - \chi(\tilde{\ell}^{(k)} \le i < \tilde{\ell}^{(k+1)}) \\
&+\chi(\tilde{s}^{(k+1)} \le i < \tilde{s}^{(k)}) - \chi(\tilde{s}^{(k)} \le i < \tilde{s}^{(k-1)}) \\
&=\chi(\dot{\tilde{\ell}}^{(k-1)} \le i < \dot{\tilde{\ell}}^{(k)}) - \chi(\dot{\tilde{\ell}}^{(k)} \le i < \dot{\tilde{\ell}}^{(k+1)}) \\
&+\chi(\dot{\tilde{s}}^{(k+1)} \le i < \dot{\tilde{s}}^{(k)}) - \chi(\dot{\tilde{s}}^{(k)} \le i < \dot{\tilde{s}}^{(k-1)}).
\end{aligned} \tag{1.37}$$

Consider the functions

$$\begin{aligned}
\Delta_i^{(k)} &= \chi(\tilde{\ell}^{(k)} \le i) - \chi(\dot{\tilde{\ell}}^{(k)} \le i) & \nabla_i^{(k)} &= \chi(\tilde{s}^{(k)} \le i) - \chi(\dot{\tilde{s}}^{(k)} \le i) \\
b_i^{-(k)} &= \chi(m_i^{(k+1)} > 0)\Delta_i^{(k)} & c_i^{-(k)} &= \chi(m_i^{(k+1)} > 0)\nabla_i^{(k)} \\
b_i^{=(k)} &= \chi(m_i^{(k)} > 0)\Delta_i^{(k)} & c_i^{=(k)} &= \chi(m_i^{(k)} > 0)\nabla_i^{(k)} \\
b_i^{+(k)} &= \chi(m_i^{(k-1)} > 0)\Delta_i^{(k)} & c_i^{+(k)} &= \chi(m_i^{(k-1)} > 0)\nabla_i^{(k)}.
\end{aligned}$$

For parts i that occur in $\nu^{(k)}$, (1.37) is implied by the following two equations:

$$b_i^{-(k-1)} - b_i^{=(k)} = b_i^{=(k)} - b_i^{+(k+1)}, \tag{1.38}$$

$$c_i^{-(k-1)} - c_i^{=(k)} = c_i^{=(k)} - c_i^{+(k+1)}. \tag{1.39}$$

It will be shown that

$$b_i^{-(k)} = b_i^{=(k)} = b_i^{+(k)} = 0 \tag{1.40}$$

$$c_i^{-(k)} = c_i^{=(k)} = c_i^{+(k)} = 0 \tag{1.41}$$

for unselected strings in $\nu^{(k+1)}$, $\nu^{(k)}$ and $\nu^{(k-1)}$, respectively. For cases I.(ℓa), I.(ℓb)(2), II.(1-3) equation (1.40) is true since $\tilde{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$. Similarly for cases I.(sa), I.(sb)(2), II.(2), II.(1')(2') and (3')(i) equation (1.41) is true since $\tilde{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ holds. Up to minor modifications the proof of (1.40) for cases I.(ℓb)(1) and I.(ℓb)(3) and of (1.41) for cases I.(sb)(1) and I.(sb)(3) is the same as in [1, Appendix A]. Also note that since $p_i^{(k)}(\tilde{\nu}) = p_i^{(k)}(\dot{\tilde{\nu}})$ (1.36) is equivalent to $p_i^{(k)}(\nu) - p_i^{(k)}(\dot{\nu}) = p_i^{(k)}(\tilde{\nu}) - p_i^{(k)}(\dot{\tilde{\nu}})$, which, in terms of the arguments, just means interchanging dot and tilde everywhere. Hence the proof of (1.40) for cases II.(1'-3') and I.(ℓs)(1') follows from cases II.(1-3) and I.(ℓs)(1). Similarly, the proof of (1.41) for cases II.(1-3) and I.(ℓs)(1') follows from the proof for cases II.(1'-3') and I.(ℓs)(1). Hence it remains to prove (1.40) for cases I.(ℓs)(1),(2) and (1.41) for cases I.(ℓs)(1),(2) and II.(3')(ii).

Unselected strings, (1.40)**.** Let us first focus on (1.40) in case I.(ℓs)(1). Note that $\Delta_i^{(k)} = \chi(\ell \le i < \ell'')$ and by the proof of case I.(ℓs)(1) $m_j^{(k-1)} = m_j^{(k)} = m_j^{(k+1)} = 0$ for $\ell < j < \ell'$ and $\ell' < j < \ell''$. By the proof of case I.(ℓs)(1) we have $m_{\ell'}^{(k+1)} = 2$ and $\dot{\ell}^{(k+1)} = \dot{s}^{(k+1)} = \ell'$ so that both strings of length ℓ' are selected. Similarly, $1 \le m_{\ell'}^{(k)} \le 2$, $\dot{s}^{(k)} = \ell'$ and $\dot{\ell}^{(k)} = \ell'$ if $m_{\ell'}^{(k)} = 2$. Hence again all strings of length ℓ' are selected in $\nu^{(k)}$. Finally $0 \le m_{\ell'}^{(k-1)} \le 2$. If $m_{\ell'}^{(k-1)} = 2$, then by the proof of lemma 1.1 case I.(ℓs)(1) holds at $k-1$ and $\dot{\ell}^{(k-1)} = \dot{s}^{(k-1)} = \ell'$. If $m_{\ell'}^{(k-1)} = 1$, then case I.(ℓs)(1) holds at $k-1$ for the first time and $\dot{s}^{(k-1)} = \ell'$. Hence again, all strings of length ℓ' in $(\nu, J)^{(k)}$ are selected. This implies that

$$\begin{aligned} b_i^{-(k)} &= \chi(i=\ell)\chi(m_i^{(k+1)} > 0) \\ b_i^{=(k)} &= \chi(i=\ell)\chi(m_i^{(k)} > 0) \\ b_i^{+(k)} &= \chi(i=\ell)\chi(m_i^{(k-1)} > 0). \end{aligned} \tag{1.42}$$

Note that either $\ell = \ell'$, in which case the above arguments already show that all strings are selected, or $\ell < \ell'$ and case I.(ℓs)(1) occurs at k for the first time. In the latter case $m_\ell^{(k)} = 1$ and $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \ell$ so that the string of length ℓ in $\nu^{(k)}$ is selected. If $\ell < \ell'$ and case I.(ℓs)(1) holds at k for the first time, equation (1.11) must hold and hence by (1.15) $m_\ell^{(k+1)} = 0$ so that $b_i^{-(k)} = 0$ for all unselected strings i. The proof that $b_i^{+(k)} = 0$ for all unselected strings i is very similar to the proof [1, Appendix A, Unselected strings, case 3].

Next consider (1.40) for the case I.(ℓs)(2). In this case $\Delta_i^{(k)} = \chi(\ell \le i < \ell'')$ and $m_j^{(k-1)} = m_j^{(k)} = m_j^{(k+1)} = 0$ for $\ell < j < \ell'$ and $\ell' < j < \ell''$. By the same arguments as in case I.(ℓs)(1) all existing strings of length ℓ' are selected. Hence (1.42) holds. Again either $\ell = \ell'$, in which case the previous arguments already show that all strings are selected, or $\ell < \ell'$ and case I.(ℓs)(2) occurs at k for the first time. In the latter case $m_\ell^{(k)} = 1$ and $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \ell$ so that the string of length ℓ in $\nu^{(k)}$ is selected. If $\ell < \ell'$ and case I.(ℓs)(2) holds at k for the first time, equation (1.11) must hold and hence by (1.15) $m_\ell^{(k+1)} = 0$ so that $b_i^{-(k)} = 0$ for all unselected strings i. The proof that $b_i^{+(k)} = 0$ for all unselected strings i is very similar to the proof [1, Appendix A, Unselected strings, case 3].

Unselected strings, (1.41)**.** Consider (1.41) for the case I.(ℓs)(1). We have $\tilde{s}^{(k)} = \tilde{\tilde{s}}^{(k)}$ so that $\nabla_i^{(k)} = 0$ unless $\tilde{\tilde{\ell}}^{(k)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \ell''$, $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell'''$, $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k)} = 1$ and $m_{\ell''}^{(k+1)} = 2$ if case I.(ℓs)(1) does not hold at $k-1$. In the former case (1.41) holds. In the latter case $\nabla_i^{(k)} = \chi(\ell'' \leq i < \ell''')$ and $m_j^{(k-1)} = m_j^{(k)} = m_j^{(k+1)} = 0$ for $\ell'' < j < \ell'''$. Hence

$$
\begin{aligned}
c_i^{-(k)} &= \chi(i = \ell'')\chi(m_{\ell''}^{(k+1)} > 0) \\
c_i^{=(k)} &= \chi(i = \ell'')\chi(m_{\ell''}^{(k)} > 0) \\
c_i^{+(k)} &= \chi(i = \ell'')\chi(m_{\ell''}^{(k-1)} > 0) = 0.
\end{aligned}
$$

Since $m_{\ell''}^{(k)} = 1$ and $\tilde{s}^{(k)} = \ell''$ the string of length ℓ'' is selected. Similarly $m_{\ell''}^{(k)} = 2$, $\tilde{s}^{(k+1)} = \ell''$ and either $\tilde{\tilde{s}}^{(k+1)} = \ell''$ if case I.(ℓs)(1) holds at $k+1$ or $\tilde{\ell}^{(k+1)} = \ell''$ otherwise. In either case both strings of length ℓ'' are selected in $\nu^{(k+1)}$. This proves (1.41).

Next consider (1.41) for the case I.(ℓs)(2). In this case $\nabla_i^{(k)} = \chi(\ell' \leq i < \ell''')$ and by the proof of lemma 1.1 $m_j^{(k-1)} = m_j^{(k)} = m_j^{(k+1)} = 0$ for $\ell' < j < \ell''$ and $\ell'' < j < \ell'''$. The strings of lengths ℓ' and ℓ'' in $\nu^{(k+1)}$ are all selected since by the proof of lemma 1.1 $m_{\ell'}^{(a)} = m_{\ell''}^{(a)} = 2$ for $k < a \leq n-2$, $m_{\ell'}^{(n-1)} = m_{\ell'}^{(n)} = m_{\ell''}^{(n-1)} = m_{\ell''}^{(n)} = 1$ and $\tilde{\ell}^{(k+1)} = \tilde{s}^{(k+1)} = \ell'$ and $\tilde{\tilde{\ell}}^{(k+1)} = \tilde{\tilde{s}}^{(k+1)} = \ell''$. Similarly, either $m_{\ell'}^{(k)} = 2$ and $\tilde{\ell}^{(k)} = \tilde{s}^{(k)}$ or $m_{\ell'}^{(k)} = 1$ and $\tilde{\ell}^{(k)} = \ell'$. This shows that all strings of lengths ℓ' are selected in $\nu^{(k)}$. Also, either $m_{\ell''}^{(k)} = 2$ and $\tilde{\tilde{\ell}}^{(k)} = \tilde{\tilde{s}}^{(k)}$ or $m_{\ell''}^{(k)} = 1$ and $\tilde{\tilde{\ell}}^{(k)} = \ell'$. This shows that all strings of lengths ℓ'' are selected in $\nu^{(k)}$. To show that all strings of length ℓ' in $\nu^{(k-1)}$ are selected, observe that either $m_{\ell'}^{(k-1)} = 0$, $m_{\ell'}^{(k-1)} = 1$ and $\tilde{s}^{(k-1)} = \ell'$ or $m_{\ell'}^{(k-1)} = 2$ and $\tilde{\ell}^{(k-1)} = \tilde{s}^{(k-1)} = \ell'$. Similarly, to show that all strings of length ℓ'' in $\nu^{(k-1)}$ are selected, observe that either $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k-1)} = 1$ and $\tilde{\tilde{\ell}}^{(k-1)} = \ell''$ or $m_{\ell''}^{(k-1)} = 2$ and $\tilde{\tilde{\ell}}^{(k-1)} = \tilde{\tilde{s}}^{(k-1)} = \ell''$.

Finally consider (1.41) for the case II.(3')(ii). Set $\ell = \tilde{\ell}^{(k)} = \dot{s}^{(k)}$, $\ell' = \tilde{s}^{(k)} = \tilde{\tilde{\ell}}^{(k)}$ and $\ell'' = \dot{\tilde{s}}^{(k)} = \tilde{\tilde{s}}^{(k)}$. From the proof of lemma 1.1 it follows that $m_{\ell'}^{(k-1)} = 0$, $m_{\ell'}^{(k)} = 1$, $m_{\ell'}^{(k+1)} = 2$ and $m_j^{(k-1)} = m_j^{(k)} = m_j^{(k+1)} = 0$ for $\ell' < j < \ell''$. Since $\nabla_i^{(k)} = \chi(\ell' \leq i < \ell'')$ we have

$$
\begin{aligned}
c_i^{-(k)} &= \chi(i = \ell')\chi(m_{\ell'}^{(k+1)} > 0) \\
c_i^{=(k)} &= \chi(i = \ell')\chi(m_{\ell'}^{(k)} > 0) \\
c_i^{+(k)} &= \chi(i = \ell')\chi(m_{\ell'}^{(k-1)} > 0) = 0.
\end{aligned}
$$

The single string of length ℓ' in $\nu^{(k)}$ is selected, so that $c_i^{=(k)} = 0$ for all unselected strings i. Furthermore, if case II.(3')(i) holds at $k+1$, then $\ell' = \tilde{s}^{(k+1)} = \tilde{\tilde{s}}^{(k+1)} = \tilde{\tilde{\ell}}^{(k+1)}$ so that both strings of length ℓ' in $\nu^{(k+1)}$ are selected. Otherwise $\tilde{\ell}^{(k+1)} = \ell' = \tilde{s}^{(k+1)}$ and again both strings of length ℓ' in $\nu^{(k+1)}$ are selected. □

References

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