Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Virtual crystal structure on rigged configurations

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ABSTRACT. Rigged configurations are combinatorial objects originating from the Bethe Ansatz, that label highest weight crystal elements. In this note a new *unrestricted* set of rigged configurations is introduced by constructing a crystal structure on the set of rigged configurations.

RÉSUMÉ. Les configurations gréées sont des objets combinatoires inspirés par l'ansatz de Bethe, et qui sont en correspondence avec les éléments cristallins de plus haut poids. Dans cette note, nous introduisons le concept de "configurations gréées généralisées", en construisant une structure cristalline dans l'espace des configurations gréées.

1. Introduction

This note is based on preprint [**33**] which gives a crystal structure on rigged configurations for all simply-laced types. Here we use the virtual crystal method [**29, 30**] to extend these results to nonsimply-laced types.

There are (at least) two main approaches to solvable lattice models and their associated quantum spin chains: the Bethe Ansatz [**6**] and the corner transfer matrix method [**5**].

In his 1931 paper [**6**], Bethe solved the Heisenberg spin chain based on the string hypothesis which asserts that the eigenvalues of the Hamiltonian form certain strings in the complex plane as the size of the system tends to infinity. The Bethe Ansatz has been applied to many models to prove completeness of the Bethe vectors. The eigenvalues and eigenvectors of the Hamiltonian are indexed by rigged configurations. However, numerical studies indicate that the string hypothesis is not always true [**2**].

The corner transfer matrix (CTM) method, introduced by Baxter [**5**], labels the eigenvectors by one-dimensional lattice paths. These lattice paths have a natural interpretation in terms of Kashiwara's crystal base theory [**16, 17**], namely as highest weight crystal elements in a tensor product of finite-dimensional crystals.

Even though neither the Bethe Ansatz nor the corner transfer matrix method are mathematically rigorous, they suggest the existence of a bijection between the two index sets, namely rigged configurations on the one hand and highest weight crystal paths on the other (see Figure 1). For the special case when the spin chain is defined on $V_{(\mu_1)} \otimes V_{(\mu_2)} \otimes \cdots \otimes V_{(\mu_k)}$, where $V_{(\mu_i)}$ is the irreducible $\mathrm{GL}(n)$ representation indexed by the partition (μ_i) for $\mu_i \in \mathbb{N}$, a bijection between rigged configurations and semi-standard Young tableaux was given by Kerov, Kirillov and Reshetikhin [**21, 22**]. This bijection was proven and extended to the case when the (μ_i) are any sequence of rectangles in [**25**]. The bijection has many amazing properties. For example it takes the cocharge statistics cc defined on rigged configurations to the coenergy statistics D defined on crystals.

Rigged configurations and crystal paths also exist for other types. In [**14, 15**] the existence of Kirillov–Reshetikhin crystals $B^{r,s}$ was conjectured, which can be naturally associated with the dominant weight $s\Lambda_r$ where s is a positive integer and Λ_r is the r-th fundamental weight of the underlying algebra of finite type. For a tensor product of Kirillov–Reshetikhin crystals $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ and a dominant weight Λ let $\overline{\mathcal{P}}(B,\Lambda)$ be the set of all highest weight elements of weight Λ in B. In the same papers [**14, 15**], fermionic formulas $\overline{M}(L,\Lambda)$ for the one-dimensional configuration sums $\overline{X}(B,\Lambda) := \sum_{b\in\overline{\mathcal{P}}(B,\Lambda)} q^{D(b)}$ were conjectured. The fermionic formulas admit a combinatorial interpretation in terms of the set of rigged configurations $\overline{\mathrm{RC}}(L,\Lambda)$, where L is the multiplicity array of B. A statistic

2000 *Mathematics Subject Classification.* Primary 17B37, 82B23, 05A15; Secondary 05E99, 81R50.
Key words and phrases. crystal bases, rigged configurations, Kashiwara operators.
Date: November 2005.
Partially supported by NSF grant DMS-0200774 and DMS-0501101.

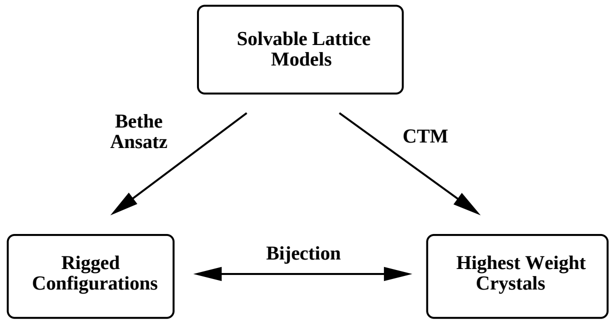


FIGURE 1. Schematic origin of rigged configurations and crystal paths

preserving bijection $\Phi : \overline{\mathcal{P}}(B,\Lambda) \to \overline{\mathrm{RC}}(L,\Lambda)$ has been proven in various cases [**25, 28, 32, 35**] which implies the following identity

$$\overline{X}(B,\Lambda) := \sum_{b\in\overline{\mathcal{P}}(B,\Lambda)} q^{D(b)} = \sum_{(\nu,J)\in\overline{\mathrm{RC}}(L,\Lambda)} q^{\mathrm{cc}(\nu,J)} =: \overline{M}(L,\Lambda). \tag{1.1}$$

Since the sets in (1.1) are finite, these are polynomials in q. When $B = B^{1,s_k}\otimes\cdots\otimes B^{1,s_1}$ of type A, they are none other than the Kostka–Foulkes polynomials.

Rigged configurations corresponding to highest weight crystal paths are only the tip of an iceberg. In this note we extend the definition of rigged configurations to all crystal elements by the explicit construction of a crystal structure on the set of *unrestricted* rigged configurations (see Definition 4.1). For simply-laced types, the proof is given in [**32**] and uses Stembridge's local characterization of simply-laced crystals [**37**]. For nonsimply-laced algebras, we show here how to apply the method of virtual crystals [**29, 30**] to construct the crystal operators on rigged configurations.

The equivalence of the crystal structures on rigged configurations and crystal paths together with the correspondence for highest weight vectors yields the equality of generating functions in analogy to (1.1) (see Theorem 4.10 and Corollary 4.11). Denote the unrestricted set of paths and rigged configurations by $\mathcal{P}(B,\Lambda)$ and $\mathrm{RC}(L,\Lambda)$, respectively. The corresponding generating functions $X(B,\Lambda) = M(L,\Lambda)$ are unrestricted generalized Kostka polynomials or q-supernomial coefficients. A direct bijection $\Phi : \mathcal{P}(B,\Lambda) \to \mathrm{RC}(L,\Lambda)$ for type A along the lines of [**25**] is constructed in [**7, 8**].

Rigged configurations are closely tied to fermionic formulas. Fermionic formulas are explicit expressions for the partition function of the underlying physical model which reflect their particle structure. For more details regarding the background of fermionic formulas see [**14, 19, 20**]. For type A we obtain an explicit characterization of the unrestricted rigged configurations in terms of lower bounds on quantum numbers which yields a new fermionic formula for unrestricted Kostka polynomials of type A. Surprisingly, this formula is different from the fermionic formulas in [**13, 18**] obtained in the special cases of $B = B^{1,s_k}\otimes\cdots\otimes B^{1,s_1}$ and $B = B^{r_k,1}\otimes\cdots\otimes B^{r_1,1}$. The rigged configurations corresponding to the fermionic formulas of [**13, 18**] were related to ribbon tableaux and the cospin generating functions of Lascoux, Leclerc, Thibon [**26, 27**] in reference [**31**]. To distinguish these rigged configurations from the ones introduced in this paper, let us call them ribbon rigged configurations.

The Lascoux–Leclerc–Thibon (LLT) polynomials [**26, 27**] have recently made their debut in the theory of Macdonald polynomials in the seminal paper by Haiman, Haglund, Loehr [**9**]. The main obstacle in obtaining a combinatorial formula for the Macdonald–Kostka polynomials is the Schur positivity of certain LLT polynomials. A related problem is the conjecture of Kirillov and Shimozono [**24**] that the cospin generating function of ribbon tableaux equals the generalized Kostka polynomial. A possible avenue to prove this conjecture would be a direct bijection between the unrestricted rigged configurations of this paper and ribbon rigged configurations.

One of the motivations for considering unrestricted rigged configurations was Takagi's work [**38**] on the inverse scattering transform, which provides a bijection between states in the $\mathfrak{sl}_2$ box ball system and rigged configurations. In this setting rigged configurations play the role of action-angle variables. Box ball systems can be produced from crystals of solvable lattice models for algebras other than $\mathfrak{sl}_2$ [**10, 11, 12**]. The inverse scattering transform can be generalized to the $\mathfrak{sl}_n$ case [**23**], which should give a box-ball interpretation of the unrestricted rigged configurations presented here.

Another motivation for the study of unrestricted configuration sums, fermionic formulas and associated rigged configurations is their appearance in generalizations of the Bailey lemma [**3, 39**]. The Andrews–Bailey construction [**1,**

4] relies on an iterative transformation property of the q-binomial coefficient, which is one of the simplest unrestricted configuration sums, and can be used to prove infinite families of Rogers–Ramanujan type identities. The explicit formulas provided in this paper might trigger further progress towards generalizations to higher-rank or other types of the Andrews–Bailey construction.

The paper is organized as follows. In Section 2 we review basics about crystal bases and virtual crystals. In Section 3 we define rigged configurations. The new crystal structure on rigged configurations is presented in section 4. Section 5 is devoted to type A, where we give an explicit characterization of the unrestricted rigged configurations, a fermionic formula for unrestricted Kostka polynomials, and the affine crystal structure.

2. Crystals

2.1. Axiomatic definition. Kashiwara [**16, 17**] introduced a **crystal** as an edge-colored directed graph satisfying a simple set of axioms. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with associated root, coroot and weight lattices $Q, Q^\vee, P$. Let I be the index set of the Dynkin diagram and denote the simple roots, simple coroots and fundamental weights by α_i, h_i and Λ_i ($i \in I$), respectively. There is a natural pairing $\langle \cdot\,,\,\cdot \rangle : Q^\vee \otimes P \to \mathbb{Z}$ defined by $\langle h_i\,,\,\Lambda_j\rangle = \delta_{ij}$.

The vertices of the crystal graph are elements of a set B. The edges of the crystal graph are colored by the index set I. A P-weighted I-crystal satisfies the following properties:

(1) Fix an $i \in I$. If all edges are removed except those colored i, the connected components are finite directed linear paths called the i-**strings** of B. Given $b \in B$, define $f_i(b)$ (resp. $e_i(b)$) to be the vertex following (resp. preceding) b in its i-string; if there is no such vertex, declare $f_i(b)$ (resp. $e_i(b)$) to be undefined. Define $\varphi_i(b)$ (resp. $\varepsilon_i(b)$) to be the number of arrows from b to the end (resp. beginning) of its i-string.
(2) There is a function $\mathrm{wt} : B \to P$ such that $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ and $\varphi_i(b) - \varepsilon_i(b) = \langle h_i\,,\,\mathrm{wt}(b)\rangle$.

2.2. Virtual crystals. There exist natural inclusions of affine Lie algebras as indicated in Figures 2 and 3. Even though these embeddings do not carry over to the corresponding quantum algebras, it is expected that such embeddings exist for crystals. Note that every affine algebra can be embedded into one of type $A^{(1)}, D^{(1)}$ and $E^{(1)}$ which are the untwisted affine algebras whose canonical simple Lie subalgebra is simply-laced. Crystal embeddings corresponding to $C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$ have been studied in [**29**], whereas the crystal embeddings $B_n^{(1)}, A_{2n-1}^{(2)} \hookrightarrow D_{n+1}^{(1)}$ have been established in [**30**].

Consider an embedding of the affine algebra with Dynkin diagram X into one with diagram Y. We consider a graph automorphism σ of Y that fixes the 0 node. For type $A_{2n-1}^{(1)}$, $\sigma(i) = 2n - i \pmod{2n}$. For type $D_{n+1}^{(1)}$ the automorphism interchanges the nodes n and $n+1$ and fixes all other nodes. There is an additional automorphism for type $D_4^{(1)}$, namely, the cyclic permutation of the nodes 1,2 and 3. For type $E_6^{(1)}$ the automorphism exchanges nodes 1 and 5 and nodes 2 and 4. In Figures 2 and 3 the automorphism σ is illustrated pictorially by arrows.

Let I^X and I^Y be the vertex sets of the diagrams X and Y respectively, I^Y/σ the set of orbits of the action of σ on I^Y, and $\iota : I^X \to I^Y/\sigma$ a bijection which preserves edges and sends 0 to 0.

EXAMPLE 2.1.
If X is one of $C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ and $Y = A_{2n-1}^{(1)}$, then $\iota(0) = 0$, $\iota(i) = \{i, 2n-i\}$ for $1 \le i < n$ and $\iota(n) = n$.
If $X = B_n^{(1)}$ or $A_{2n-1}^{(2)}$ and $Y = D_{n+1}^{(1)}$, then $\iota(i) = i$ for $i < n$ and $\iota(n) = \{n, n+1\}$.
If X is $D_4^{(3)}$ or $G_2^{(1)}$ and $Y = D_4^{(1)}$, then $\iota(0) = 0$, $\iota(1) = 2$ and $\iota(2) = \{1,3,4\}$.
If X is $E_6^{(2)}$ or $F_4^{(1)}$ and $Y = E_6^{(1)}$, then $\iota(0) = 0$, $\iota(1) = 1$, $\iota(2) = 3$, $\iota(3) = \{2,4\}$ and $\iota(4) = \{1,5\}$.

To describe the embedding we endow the bijection ι with additional data. For each $i \in I^X$ we shall define a multiplication factor γ_i that depends on the location of i with respect to a distinguished arrow (multiple bond) in X. Removing the arrow leaves two connected components. The factor γ_i is defined as follows:

(1) Suppose X has a unique arrow.
    (a) Suppose the arrow points towards the component of 0. Then $\gamma_i = 1$ for all $i \in I^X$.
    (b) Suppose the arrow points away from the component of 0. Then γ_i is the order of σ for i in the component of 0 and is 1 otherwise.
(2) Suppose X has two arrows. Then $\gamma_i = 1$ for $1 \le i \le n-1$. For $i \in \{0, n\}$, $\gamma_i = 2$ (which is the order of σ) if the arrow incident to i points away from it and is 1 otherwise.

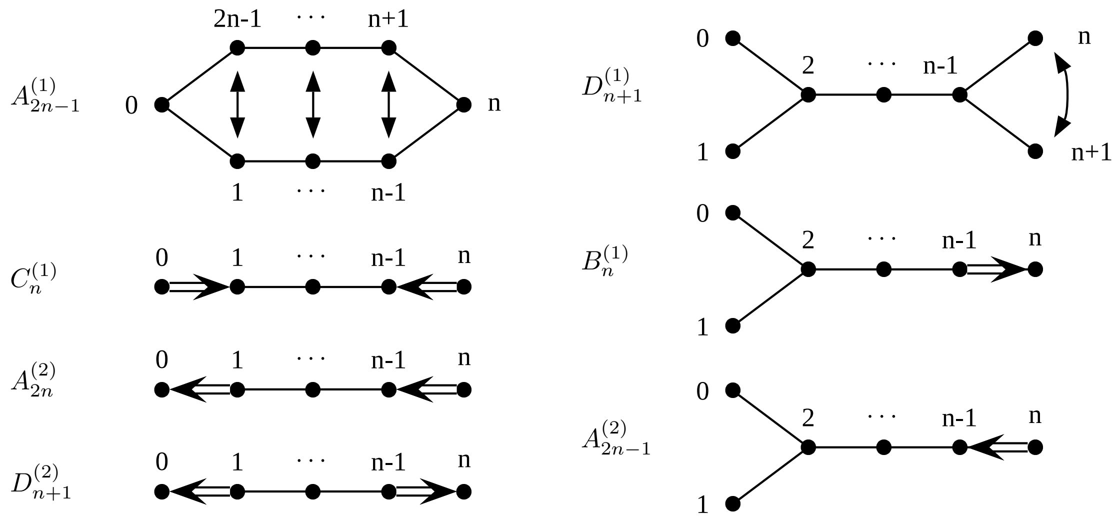


FIGURE 2. Embeddings $C^{(1)}_n, A^{(2)}_{2n}, D^{(2)}_{n+1} \hookrightarrow A^{(1)}_{2n-1}$ and $B^{(1)}_n, A^{(2)}_{2n-1} \hookrightarrow D^{(1)}_{n+1}$

EXAMPLE 2.2. The values of γ_i are summarized in the following table:

X	
$A^{(2)}_{2n-1}$ $D^{(3)}_4$ $E^{(2)}_6$	$\gamma_i = 1$ for all i
$B^{(1)}_n$	$\gamma_i = 2$ for $0 \le i \le n-1$ $\gamma_n = 1$
$G^{(1)}_2$	$\gamma_i = 3$ for $i = 0, 1$ $\gamma_2 = 1$
$F^{(1)}_4$	$\gamma_i = 2$ for $i = 0, 1, 2$ $\gamma_i = 1$ for $i = 3, 4$
$C^{(1)}_n$	$\gamma_i = 1$ for $1 \le i < n$ $\gamma_0 = \gamma_n = 2$
$A^{(2)}_{2n}$	$\gamma_i = 1$ for $0 \le i < n$ $\gamma_n = 2$
$D^{(2)}_{n+1}$	$\gamma_i = 1$ for all i

The embedding $\Psi : P^X \to P^Y$ of weight lattices is defined by

$$\Psi(\Lambda^X_i) = \gamma_i \sum_{j \in \iota(i)} \Lambda^Y_j .$$

Let $\widehat{V}$ be a Y-crystal. We define the virtual crystal operators $\widehat{e}_i, \widehat{f}_i$ for $i \in I^X$ as the composites of Y-crystal operators f_j, e_j given by

$$\widehat{f}_i = \prod_{j \in \iota(i)} f_j^{\gamma_i} \qquad \text{and} \qquad \widehat{e}_i = \prod_{j \in \iota(i)} e_j^{\gamma_i}. \tag{2.1}$$

These are designed to simulate X-crystal operators f_i, e_i for $i \in I^X$. The type Y operators on the right hand side, may be performed in any order, since distinct nodes $j, j' \in \iota(i)$ are not adjacent in Y and thus their corresponding raising and lowering operators commute.

A **virtual crystal** is a pair $(V, \widehat{V})$ such that:

(1) $\widehat{V}$ is a Y-crystal.
(2) $V \subset \widehat{V}$ is closed under $\widehat{e}_i, \widehat{f}_i$ for $i \in I^X$.

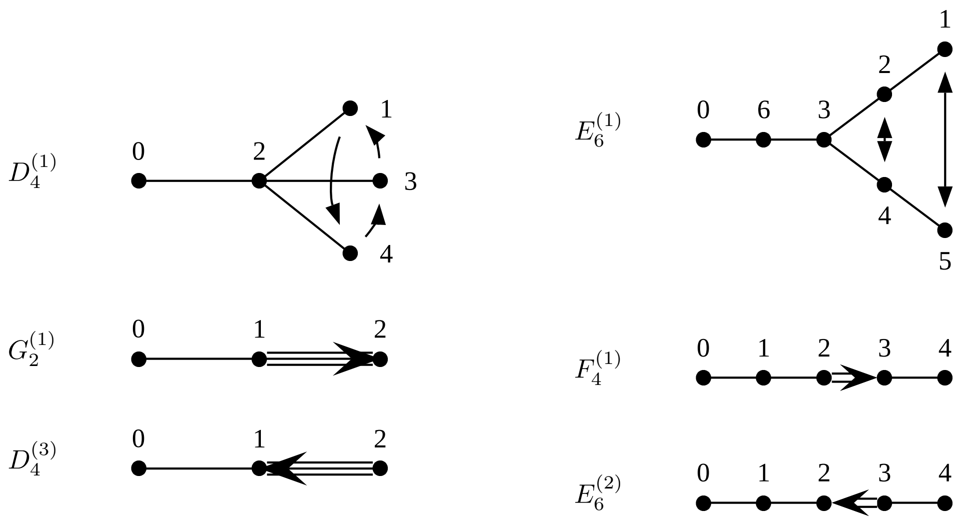


FIGURE 3. Embeddings $G_2^{(1)}, D_4^{(3)} \hookrightarrow D_4^{(1)}$ and $F_4^{(1)}, E_6^{(2)} \hookrightarrow E_6^{(1)}$

(3) There is an X-crystal B and an X-crystal isomorphism $\Psi : B \to V$ such that e_i, f_i correspond to $\widehat{e}_i, \widehat{f}_i$.

Sometimes by abuse of notation, V will be referred to as a virtual crystal.

Let us define the Y-crystal

$$\widehat{V}^{r,s} = \bigotimes_{j \in \iota(r)} B_Y^{j,\gamma_r s}$$

except for $A_{2n}^{(2)}$ and $r = n$ in which case $\widehat{V}^{n,s} = B_Y^{n,s} \otimes B_Y^{n,s}$. Denote by $u(\widehat{V}^{r,s})$ the extremal vector of weight $\Psi(s\Lambda_r)$ in $\widehat{V}^{r,s}$.

DEFINITION 2.3. Let $V^{r,s}$ be the subset of $\widehat{V}^{r,s}$ generated from $u(\widehat{V}^{r,s})$ using the virtual crystal operators $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in I^X$.

CONJECTURE 2.4. [**30**, Conjecture 3.7] *There is an isomorphism of X-crystals $\Psi : B_X^{r,s} \cong V^{r,s}$ such that e_i and f_i correspond to $\widehat{e}_i$ and $\widehat{f}_i$ respectively, for all $i \in I^X$.*

In [**29**] Conjecture 2.4 is proved for embeddings $C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$ and $s = 1$. In [**30**] Conjecture 2.4 is proved for all nonexceptional types when $r = 1$.

3. Rigged configurations

In this section we define rigged configurations for all affine Kac–Moody algebras. Type $A_{2n}^{(2)}$ requires some special treatment. We need the variant $\widetilde{\gamma}_a$ of the multiplication factor γ_a which is $\widetilde{\gamma}_a = \gamma_a$ except for $A_{2n}^{(2)}$ and $a = n$ when $\widetilde{\gamma}_n = 1$. Also set $\widetilde{\alpha}_a = \alpha_a$ for all $a \in I$ except for type $A_{2n}^{(2)}$ in which case $\widetilde{\alpha}_a$ are the simple roots of type B_n.

Let $L = (L_i^{(a)})_{(a,i) \in \mathcal{H}}$ be an array of nonnegative integers where $\mathcal{H} = \{1, 2, \dots, n\} \times \mathbb{Z}_{>0}$, called the multiplicity array, where n is the rank of the underlying algebra and Λ a weight. Then an (L, Λ)-configuration is an array $m = (m_i^{(a)})_{(a,i) \in \mathcal{H}}$ such that

$$\sum_{(a,i) \in \mathcal{H}} i\, m_i^{(a)} \widetilde{\alpha}_a = \sum_{(a,i) \in \mathcal{H}} i\, L_i^{(a)} \Lambda_a - \Lambda \tag{3.1}$$

except for type $A_{2n}^{(2)}$. In this case the right hand side should be replaced by $\iota(\mathrm{r.h.s})$ where ι is a $\mathbb{Z}$-linear map from the weight lattice of type C_n to the weight lattice of type B_n such that

$$\iota(\Lambda_a^C) = \begin{cases} \Lambda_a^B & \text{for } 1 \le a < n \\ 2\Lambda_a^B & \text{for } a = n. \end{cases}$$

The vacancy numbers of a given configuration are defined as

$$p_i^{(a)} = \sum_{(b,j)\in\mathcal{H}} -\frac{2(\alpha_a \mid \alpha_b)}{\gamma_b(\alpha_a \mid \alpha_a)} \min(\widetilde{\gamma}_a i, \widetilde{\gamma}_b j) m_j^{(b)} + \sum_{j\ge 0} \min(i,j) L_j^{(a)}. \tag{3.2}$$

An (L,Λ)-configuration is called **admissible** if $p_i^{(a)} \ge 0$ for all $(a,i)\in\mathcal{H}$. The set of admissible (L,Λ)-configurations is denoted by $\overline{\mathrm{C}}(L,\Lambda)$.

A **rigged configuration** is a pair (m,J) where $m = (m_i^{(a)})_{(a,i)\in\mathcal{H}}$ is an admissible (L,Λ)-configuration and $J = (J_i^{(a)})_{(a,i)\in\mathcal{H}}$ is a matrix of partitions such that the partition $J_i^{(a)}$ is contained in a rectangle of size $m_i^{(a)} \times p_i^{(a)}$. The set of rigged configurations for fixed L and Λ is denoted by $\overline{\mathrm{RC}}(L,\Lambda)$.

Rigged configurations can also be represented as a sequence of partitions such that each part of each partition is labeled or "rigged" by a number. Let $\nu = (\nu^{(1)},\nu^{(2)},\dots,\nu^{(n)})$ be the sequence of partitions obtained from $m = (m_i^{(a)})$ as follows. Let $m_i^{(a)}(\nu)$ be the number of parts in $\nu^{(a)}$ of size i. Then ν is determined by requiring that

$$m_{\widetilde{\gamma}_a i}^{(a)}(\nu) = m_i^{(a)} \quad \text{and} \quad m_j^{(a)}(\nu) = 0 \quad \text{for } j \notin \widetilde{\gamma}_a\mathbb{Z}.$$

The vacancy number $P_i^{(a)}(\nu)$ for each part i of $\nu^{(a)}$ is then

$$P_i^{(a)}(\nu) = \sum_{b\in I} -\frac{2(\alpha_a \mid \alpha_b)}{\gamma_b(\alpha_a \mid \alpha_a)} Q_i(\nu^{(b)}) + \sum_{j\ge 0} \min(\frac{i}{\widetilde{\gamma}_a}, j) L_j^{(a)},$$

where $Q_i(\rho)$ is the number of boxes in the first i columns of the partition ρ. The relation to $p_i^{(a)}$ is

$$p_i^{(a)} = P_{\widetilde{\gamma}_a i}^{(a)}(\nu).$$

A tuple (i,x) where i is a part of $\nu^{(a)}$ and x is a part of $J_i^{(a)}$ is called a string of the rigged partition $(\nu,J)^{(a)}$. Here i is the length and x the label of the string. The **colabel** of a string (i,x) of $(\nu,J)^{(a)}$ is $P_i^{(a)}(\nu) - x$.

EXAMPLE 3.1. Let $\Lambda = \Lambda_1 + \Lambda_3$ of type $A_6^{(2)}$, $L_1^{(1)} = 7$ and all other $L_i^{(a)} = 0$. Then

$$(\nu, J) \quad = \quad \begin{array}{|c|l}\hline & 0\ \ 0 \\ \hline & 0\ \ 0 \\ \hline & 0\ \ 0 \\ \hline & 0\ \ 0 \\ \hline & 0\ \ 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|l}\cline{1-2} & & 0\ \ 0 \\ \cline{1-2} & \multicolumn{2}{|l}{1\ \ 1} \\ \cline{1-1} & \multicolumn{2}{|l}{1\ \ 1} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|l}\cline{1-2} & & 1\ \ 1 \\ \cline{1-2} & \multicolumn{2}{|l}{0\ \ 1} \\ \cline{1-1} \end{array} \in \overline{\mathrm{RC}}(L,\Lambda),$$

where the first number behind each part is the label and the second one is the vacancy number.

There is also a statistic called **cocharge** defined on rigged configurations. Set $t_a^\vee = \frac{|\iota(a)|\gamma_a}{\gamma_0}$. The cocharge is given by

$$\begin{aligned} \mathrm{cc}(\nu) &= \sum_{(i,a),(b,j)\in\mathcal{H}} \frac{t_a^\vee}{\gamma_b} \cdot \frac{(\alpha_a \mid \alpha_b)}{(\alpha_a \mid \alpha_a)} \min(\widetilde{\gamma}_a i, \widetilde{\gamma}_b j) m_i^{(a)} m_j^{(b)} \\ &= \frac{1}{2} \sum_{(a,i)\in\mathcal{H}} t_a^\vee m_i^{(a)} \big(\sum_{j\ge 0} \min(i,j) L_j^{(a)} - p_i^{(a)} \big) \end{aligned} \tag{3.3}$$

for a configuration ν and $\mathrm{cc}(\nu,J) = \mathrm{cc}(\nu) + |J|$ where $|J| = \sum_{(a,i)\in\mathcal{H}} t_a^\vee |J_i^{(a)}|$ is the sum of the sizes of all partitions $J_i^{(a)}$ weighted by $t_a^\vee$.

As mentioned in the introduction, rigged configurations correspond to highest weight crystal elements. Let $B^{r,s}$ be a Kirillov–Reshetikhin crystal for $(r,s)\in\mathcal{H}$ and $B = B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$. Associate to B the multiplicity array $L = (L_s^{(r)})_{(r,s)\in\mathcal{H}}$ where $L_s^{(r)}$ counts the number of tensor factors $B^{r,s}$ in B. Denote by

$$\overline{\mathcal{P}}(B,\Lambda) = \{ b \in B \mid \mathrm{wt}(b) = \Lambda,\ e_i(b) \text{ undefined for all } i \in I \}$$

the set of all highest weight elements of weight Λ in B. There is a natural statistics defined on B, called energy function or more precisely tail coenergy function $D : B \to \mathbb{Z}$ (see [**35**, Eq. (5.1)] for a precise definition).

The following theorem was proven in [**25**] for type $A_{n-1}^{(1)}$ and general $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$, in [**32**] for type $D_n^{(1)}$ and $B = B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$ and in [**35**] for type $D_n^{(1)}$ and $B = B^{1,s_k} \otimes \cdots \otimes B^{1,s_1}$.

THEOREM 3.2. [**25, 32, 35**] *For Λ a dominant weight, B as above and L the corresponding multiplicity array, there is a bijection $\overline{\Phi} : \overline{\mathcal{P}}(B,\Lambda) \to \overline{\mathrm{RC}}(L,\Lambda)$ which preserves the statistics, that is, $D(b) = \mathrm{cc}(\overline{\Phi}(b))$ for all $b \in \overline{\mathcal{P}}(B,\Lambda)$.*

Defining the generating functions

$$\overline{X}(B,\Lambda) = \sum_{b\in\overline{\mathcal{P}}(B,\Lambda)} q^{D(b)} \qquad \text{and} \qquad \overline{M}(L,\Lambda) = \sum_{(\nu,J)\in\overline{\mathrm{RC}}(L,\Lambda)} q^{\mathrm{cc}(\nu,J)}, \tag{3.4}$$

we get the immediate corollary of Theorem 3.2.

COROLLARY 3.3. [**25, 32, 35**] *Let Λ, B and L as in Theorem 3.2. Then $\overline{X}(B,\Lambda) = \overline{M}(L,\Lambda)$.*

4. Crystal structure on rigged configurations

The rigged configurations of section 3 correspond to highest weight crystal elements. In this section we introduce the set of unrestricted rigged configurations $\mathrm{RC}(L)$ by defining a crystal structure generated from highest weight vectors given by elements in $\overline{\mathrm{RC}}(L) = \bigcup_{\Lambda\in P^+} \overline{\mathrm{RC}}(L,\Lambda)$ by the Kashiwara operators e_a, f_a. For simply-laced algebras the following definition was given in [**33**, Definition 3.3]. The multiplication factors γ_a for the simply-laced case are equal to 1.

DEFINITION 4.1. Let L be a multiplicity array. Define the set of **unrestricted rigged configurations** $\mathrm{RC}(L)$ as the set generated from the elements in $\overline{\mathrm{RC}}(L)$ by the application of the operators f_a, e_a for $1 \le a \le n$ defined as follows:

(1) Define $e_a(\nu,J)$ by removing γ_a boxes from a string of length k in $(\nu,J)^{(a)}$ leaving all colabels fixed and increasing the new label by one. Here k is the length of the string with the smallest negative rigging of smallest length. If no such string exists, $e_a(\nu,J)$ is undefined.
(2) Define $f_a(\nu,J)$ by adding γ_a boxes to a string of length k in $(\nu,J)^{(a)}$ leaving all colabels fixed and decreasing the new label by one. Here k is the length of the string with the smallest nonpositive rigging of largest length. If no such string exists, add a new string of length one and label -1. If the result is not a valid unrestricted rigged configuration $f_a(\nu,J)$ is undefined.

EXAMPLE 4.2. For (ν,J) of Example 3.1 we have

$$f_1(\nu,J) \quad = \quad \begin{array}{l} \square\square\ \ {-1}\ {-1} \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \end{array} \qquad \begin{array}{l} \square\square\ \ 1\ \ 1 \\ \square\ \ 1\ \ 1 \\ \square\ \ 1\ \ 1 \end{array} \qquad \begin{array}{l} \square\square\ \ 1\ \ 1 \\ \square\ \ 0\ \ 1 \end{array}$$

and

$$f_3(\nu,J) \quad = \quad \begin{array}{l} \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \\ \square\ \ 0\ \ 0 \end{array} \qquad \begin{array}{l} \square\square\ \ 1\ \ 1 \\ \square\ \ 1\ \ 1 \\ \square\ \ 1\ \ 1 \end{array} \qquad \begin{array}{l} \square\square\square\ \ {-1}\ {-1} \\ \square\square\ \ 0\ \ 0 \end{array}.$$

THEOREM 4.3. *The operators e_a, f_a of Definition 4.1 are the Kashiwara crystal operators.*

For simply-laced algebras Theorem 4.3 was proven in [**33**] by using the local characterization of simply-laced crystals given by Stembridge [**37**]. In the following we show that, assuming that the virtual crystal embeddings of section 2.2 hold, Theorem 4.3 is also true for the nonsimply-laced algebras.

We define **virtual rigged configurations** in analogy to virtual crystals. Here $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ is a tensor product of Kirillov-Reshetikhin crystals and $L = (L_i^{(a)})$ the corresponding multiplicity array.

DEFINITION 4.4. Let $X \hookrightarrow Y$ be one of the algebra embeddings of section 2.2, Λ a weight and B a crystal for type X. Let $(V,\widehat{V})$ be the virtual Y-crystal corresponding to B. Then $\mathrm{RC}^v(L,\Lambda)$ is the set of elements $(\widehat{\nu},\widehat{J}) \in \mathrm{RC}(\widehat{L},\Psi(\Lambda))$ such that:

(1) For all $i \in \mathbb{Z}_{>0}$, $\widehat{m}_i^{(a)} = \widehat{m}_i^{(b)}$ and $\widehat{J}_i^{(a)} = \widehat{J}_i^{(b)}$ if a and b are in the same σ-orbit in I^Y.
(2) For all $i \in \mathbb{Z}_{>0}$, $a \in I^X$, and $b \in \iota(a) \subset I^Y$, we have $\widehat{m}_j^{(b)} = 0$ if $j \notin \widetilde{\gamma}_a\mathbb{Z}$ and the parts of $\widehat{J}_i^{(b)}$ are multiples of γ_a.

THEOREM 4.5. [**30**, Theorem 4.2] *There is a bijection* $\mathrm{RC}(L,\Lambda) \to \mathrm{RC}^v(L,\Lambda)$ *sending* $(\nu, J) \mapsto (\widehat{\nu}, \widehat{J})$ *given as follows. For all* $a \in I^X$*,* $b \in \iota(a) \subset I^Y$*, and* $i \in \mathbb{Z}_{>0}$*,*

$$\widehat{m}^{(b)}_{\widetilde{\gamma}_a i} = m^{(a)}_i \qquad \textit{and} \qquad \widehat{J}^{(b)}_{\widetilde{\gamma}_a i} = \gamma_a J^{(a)}_i.$$

The cocharge changes by $\mathrm{cc}(\widehat{\nu}, \widehat{J}) = \gamma_0 \, \mathrm{cc}(\nu, J)$.

PROOF OF THEOREM 4.3. Theorem 4.3 was proved in [**33**] for the simply-laced algebras. Hence, assuming that the virtual crystal embeddings of section 2.2 hold, it suffices to check that e_a, f_a of Definition 4.1 satisfy (2.1). By Theorem 4.5 this reduces to checking that $\widehat{f}_a$ and $\widehat{e}_a$ preserve the conditions of Definition 4.4. We demonstrate this for $\widehat{f}_a$; the arguments for $\widehat{e}_a$ are analogous. Let $(\widehat{\nu}, \widehat{J}) \in \mathrm{RC}^v(L,\Lambda)$. Since f_a and f_b of Definition 4.1 for simply-laced algebras commute if $b \in \iota(a)$, point (1) of Definition 4.4 follows for $\widehat{f}_a(\widehat{\nu}, \widehat{J})$. To prove that point (2) holds, it suffices to check that if $\gamma_a > 1$, then the various applications of f_a in $\widehat{f}_a$ select the same string γ_a times. Note that for simply-laced algebras the application of f_a changes the vacancy number $\widehat{p}^{(b)}_i$ by

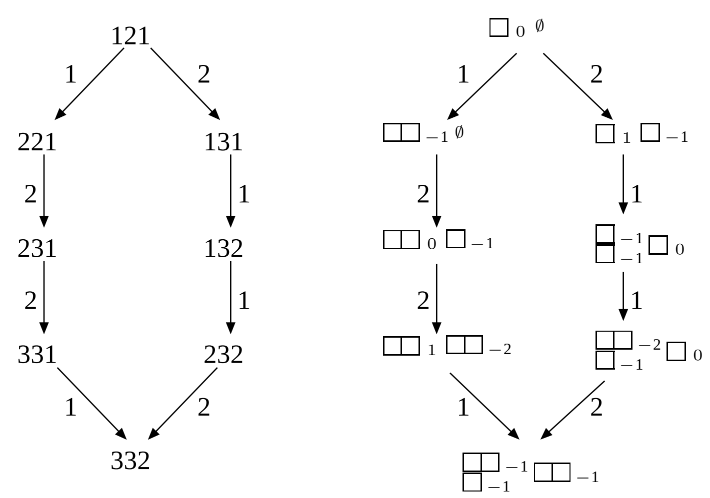
$$\widehat{p}^{(b)}_i \mapsto \widehat{p}^{(b)}_i - (\alpha_a \mid \alpha_b)\chi(i > k), \tag{4.1}$$

where k is the length of the selected string. By the definition of k (see Definition 4.1) and the fact that all riggings in the a-th rigged partition have parity γ_a by point (2) of Definition 4.4, all riggings of strings of length $i > k$ in $(\widehat{\nu}, \widehat{J})^{(a)}$ are greater or equal to $-s + \gamma_a$, where $-s$ is the smallest rigging appearing in $(\widehat{\nu}, \widehat{J})^{(a)}$. By (4.1) the riggings of length $i > k$ in $(\widehat{\nu}, \widehat{J})^{(a)}$ change by -2. Hence the smallest j such that $-s + \gamma_a - 2j \le -s - j$ is $j = \gamma_a$. This shows that γ_a applications of f_a select the same string, which in turn proves that $\widehat{f}_a(\widehat{\nu}, \widehat{J})$ satisfies the conditions of Definition 4.4. □

THEOREM 4.6. *With the same assumptions as in Theorem 3.2, the graph generated from* $(\overline{\nu}, \overline{J}) \in \overline{\mathrm{RC}}(L,\Lambda)$ *and the crystal operators* e_a, f_a *of Definition 4.1 is isomorphic to the crystal graph* $B(\Lambda)$ *of highest weight* Λ.

PROOF. For simply-laced types this was proven in [**33**, Theorem 3.7]. For nonsimply-laced types this follows from Theorems 4.3 and 4.5. □

EXAMPLE 4.7. Consider the crystal $B(\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix})$ of type A_2 in $B = (B^{1,1})^{\otimes 3}$. Here is the crystal graph in the usual labeling and the rigged configuration labeling:



THEOREM 4.8. *The cocharge* cc *as defined in* (3.3) *is constant on connected crystal components.*

PROOF. For simply-laced types this was proved in [**33**, Theorem 3.9]. For nonsimply-laced types this follows from Theorems 4.3 and 4.5. □

EXAMPLE 4.9. The cocharge of the connected component in Example 4.7 is 1.

For $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ and $\Lambda \in P$ let

$$\mathcal{P}(B,\Lambda) = \{b \in B \mid \mathrm{wt}(b) = \Lambda\}.$$

THEOREM 4.10. *Let $\Lambda \in P$, B be as in Theorem 3.2 and L the corresponding multiplicity array. Then there is a bijection $\Phi : \mathcal{P}(B,\Lambda) \to \mathrm{RC}(L,\Lambda)$ which preserves the statistics, that is, $D(b) = \mathrm{cc}(\Phi(b))$ for all $b \in \mathcal{P}(B,\Lambda)$.*

PROOF. By Theorem 3.2 there is such a bijection for the maximal elements $b \in \overline{\mathcal{P}}(B)$. By Theorems 4.6 and 4.8 this extends to all of $\mathcal{P}(B,\Lambda)$. □

Extending the definitions of (3.4) to

$$X(B,\Lambda) = \sum_{b \in \mathcal{P}(B,\Lambda)} q^{D(b)} \qquad \text{and} \qquad M(L,\Lambda) = \sum_{(\nu,J)\in \mathrm{RC}(L,\Lambda)} q^{\mathrm{cc}(\nu,J)}, \tag{4.2}$$

we obtain the corollary:

COROLLARY 4.11. *With all hypotheses of Theorem 4.10, we have $X(B,\Lambda) = M(L,\Lambda)$.*

5. Unrestricted rigged configurations for type $A^{(1)}_{n-1}$

In this section we give an explicit description of the elements in $\mathrm{RC}(L,\lambda)$ for type $A^{(1)}_{n-1}$. Generally speaking, the elements are rigged configurations where the labels lie between the vacancy number and certain lower bounds defined explicitly. This characterization will be used in section 5.2 to write down an explicit fermionic formula $M(L,\lambda)$ for the unrestricted configuration sum $X(B,\lambda)$. Section 5.3 is devoted to the affine crystal structure of $\mathrm{RC}(L,\lambda)$.

5.1. Characterization of unrestricted rigged configurations. Let $L = (L_i^{(a)})_{(a,i)\in\mathcal{H}}$ be a multiplicity array and $\lambda = (\lambda_1,\ldots,\lambda_n)$ be the n-tuple of nonnegative integers. The set of (L,λ)-configurations $\mathrm{C}(L,\lambda)$ is the set of all sequences of partitions $\nu = (\nu^{(a)})_{a\in I}$ such that (3.1) holds. As discussed in Section 3, in the usual setting a rigged configuration $(\nu,J) \in \overline{\mathrm{RC}}(L,\lambda)$ consists of a configuration $\nu \in \overline{\mathrm{C}}(L,\lambda)$ together with a double sequence of partitions $J = \{J_i^{(a)} \mid (a,i) \in \mathcal{H}\}$ such that the partition $J_i^{(a)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle. In particular this requires that $p_i^{(a)} \geq 0$. The unrestricted rigged configurations $(\nu,J) \in \mathrm{RC}(L,\lambda)$ can contain labels that are negative, that is, the lower bound on the parts in $J_i^{(a)}$ can be less than zero.

To define the lower bounds we need the following notation. Let $\lambda' = (c_1, c_2, \ldots, c_{n-1})^t$, where $c_k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$ is the length of the k-th column of λ', and let $\mathcal{A}(\lambda')$ be the set of tableaux of shape λ' such that the entries are strictly decreasing along columns, and the letters in column k are from the set $\{1,2,\ldots,c_{k-1}\}$ with $c_0 = c_1$.

EXAMPLE 5.1. For $n = 4$ and $\lambda = (0,1,1,1)$, the set $\mathcal{A}(\lambda')$ consists of the following tableaux

$$\begin{array}{|c|c|c|}\hline 3&3&2\\ \hline 2&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&3&2\\ \hline 2&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&2&2\\ \hline 2&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&3&1\\ \hline 2&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&3&1\\ \hline 2&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&2&1\\ \hline 2&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}\,.$$

REMARK 5.2. Denote by $t_{j,k}$ the entry of $t \in \mathcal{A}(\lambda')$ in row j and column k. Note that $c_k - j + 1 \leq t_{j,k} \leq c_{k-1} - j + 1$ since the entries in column k are strictly decreasing and lie in the set $\{1,2,\ldots,c_{k-1}\}$. This implies $t_{j,k} \leq c_{k-1} - j + 1 \leq t_{j,k-1}$, so that the rows of t are weakly decreasing.

Given $t \in \mathcal{A}(\lambda')$, we define the **lower bound** as

$$M_i^{(a)}(t) = -\sum_{j=1}^{c_a} \chi(i \geq t_{j,a}) + \sum_{j=1}^{c_{a+1}} \chi(i \geq t_{j,a+1}),$$

where recall that $\chi(S) = 1$ if the the statement S is true and $\chi(S) = 0$ otherwise.

Let $M,p,m \in \mathbb{Z}$ such that $m \geq 0$. A (M,p,m)-quasipartition μ is a tuple of integers $\mu = (\mu_1,\mu_2,\ldots,\mu_m)$ such that $M \leq \mu_m \leq \mu_{m-1} \leq \cdots \leq \mu_1 \leq p$. Each μ_i is called a part of μ. Note that for $M = 0$ this would be a partition with at most m parts each not exceeding p.

The following theorem shows that the set of unrestricted rigged configurations can be characterized via the lower bounds.

THEOREM 5.3. [**33**, Theorem 4.6] *Let $(\nu,J) \in \mathrm{RC}(L,\lambda)$. Then $\nu \in \mathrm{C}(L,\lambda)$ and $J_i^{(a)}$ is a $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-quasipartition for some $t \in \mathcal{A}(\lambda')$. Conversely, every (ν,J) such that $\nu \in \mathrm{C}(L,\lambda)$ and $J_i^{(a)}$ is a $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-quasipartition for some $t \in \mathcal{A}(\lambda')$ is in $\mathrm{RC}(L,\lambda)$.*

EXAMPLE 5.4. Let $n = 4$, $\lambda = (2,2,1,1)$, $L_1^{(1)} = 6$ and all other $L_i^{(a)} = 0$. Then

$$(\nu, J) \;=\; \begin{array}{l} \square\square\square \; -2\ 0 \\ \square \;\; 0\ 3 \end{array} \qquad \square\square \;\; 0\ \ 0 \qquad \square \; -1\ \text{-}1$$

is an unrestricted rigged confi guration in $\mathrm{RC}(L, \lambda)$, where we have written the parts of $J_i^{(a)}$ next to the parts of length i in partition $\nu^{(a)}$. The second number is the corresponding vacancy number $p_i^{(a)}$. This shows that the labels are indeed all weakly below the vacancy numbers. For

$$\begin{array}{|c|c|c|} \hline 4 & 4 & 1 \\ \hline 3 & 3 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array} \in \mathcal{A}(\lambda')$$

we get the lower bounds

$$\begin{array}{l} \square\square\square \; -2 \\ \square \; -1 \end{array} \qquad \square\square \;\; 0 \qquad \square \; -1,$$

which are less or equal to the riggings in (ν, J).

For type A_1 we have $\lambda = (\lambda_1, \lambda_2)$ so that $\mathcal{A} = \{t\}$ contains just the single one-column tableau of height λ_2 fi lled with the numbers $1, 2, \ldots, \lambda_2$. In this case $M_i(t) = -\sum_{j=1}^{\lambda_2} \chi(i \geq t_{j,1}) = -i$, which agrees with the fi ndings of [**38**].

The characterization of unrestricted rigged confi gurations is similar to the characterization of level-restricted rigged confi gurations [**34**, Defi nition 5.5]. Whereas the unrestricted rigged confi gurations are characterized in terms of lower bounds, for level-restricted rigged confi gurations the vacancy number has to be modifi ed according to tableaux in a certain set.

5.2. Fermionic formula. With the explicit characterization of the unrestricted rigged confi gurations of Section 5.1, it is possible to derive an explicit formula for the polynomials $M(L, \lambda)$ of (4.2).

Let $\mathcal{SA}(\lambda')$ be the set of all nonempty subsets of $\mathcal{A}(\lambda')$ and set

$$M_i^{(a)}(S) = \max\{M_i^{(a)}(t) \mid t \in S\} \qquad \text{for } S \in \mathcal{SA}(\lambda').$$

By inclusion-exclusion the set of all allowed riggings for a given $\nu \in \mathrm{C}(L, \lambda)$ is

$$\bigcup_{S \in \mathcal{SA}(\lambda')} (-1)^{|S|+1} \{J \mid J_i^{(a)} \text{ is a } (M_i^{(a)}(S), p_i^{(a)}, m_i^{(a)})\text{-quasipartition}\}.$$

The q-binomial coeffi cient $\begin{bmatrix} m+p \\ m \end{bmatrix}$, defi ned as

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p},$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$, is the generating function of partitions with at most m parts each not exceeding p. Hence the polynomial $M(L, \lambda)$ may be rewritten as

$$M(L,\lambda) = \sum_{S \in \mathcal{SA}(\lambda')} (-1)^{|S|+1} \sum_{\nu \in \mathrm{C}(L,\lambda)} q^{\mathrm{cc}(\nu) + \sum_{(a,i)\in\mathcal{H}} m_i^{(a)} M_i^{(a)}(S)} \prod_{(a,i)\in\mathcal{H}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} - M_i^{(a)}(S) \\ m_i^{(a)} \end{bmatrix} \tag{5.1}$$

called **fermionic formula**. By Corollary 4.11 this is also a formula for the unrestricted confi guration sum $X(B, \lambda)$. This formula is different from the fermionic formulas of [**13, 18**] which exist in the special case when L is the multiplicity array of $B = B^{1,s_k} \otimes \cdots \otimes B^{1,s_1}$ or $B = B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$.

5.3. The Kashiwara operators e_0 and f_0. The Kirillov–Reshetikhin crystals $B^{r,s}$ are affi ne crystals and admit the Kashiwara operators e_0 and f_0. It was shown in [**36**] that for type $A_{n-1}^{(1)}$ they can be defi ned in terms of the **promotion operator** pr as

$$e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad \text{and} \quad f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}.$$

The promotion operator is a bijection $\mathrm{pr} : B \to B$ such that the following diagram commutes for all $a \in I$

$$\begin{tikzcd} B \arrow[r, "\mathrm{pr}"] \arrow[d, "f_a"'] & B \arrow[d, "f_{a+1}"] \\ B \arrow[r, "\mathrm{pr}"'] & B \end{tikzcd} \tag{5.2}$$

and such that for every $b \in B$ the weight is rotated

$$\langle h_{a+1}\, ,\, \mathrm{wt}(pr(b))\rangle = \langle h_a\, ,\, \mathrm{wt}(b)\rangle. \tag{5.3}$$

Here subscripts are taken modulo n.

We are now going to define the promotion operator on unrestricted rigged configurations.

DEFINITION 5.5. Let $(\nu, J) \in \mathrm{RC}(L, \lambda)$. Then $\mathrm{pr}(\nu, J)$ is obtained as follows:

(1) Set $(\nu', J') = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}(\nu, J)$ where f_n acts on $(\nu, J)^{(n)} = \emptyset$.
(2) Apply the following algorithm ρ to (ν', J') λ_n times: Find the smallest singular string in $(\nu', J')^{(n)}$. Let the length be $\ell^{(n)}$. Repeatedly find the smallest singular string in $(\nu', J')^{(k)}$ of length $\ell^{(k)} \geq \ell^{(k+1)}$ for all $1 \leq k < n$. Shorten the selected strings by one and make them singular again.

EXAMPLE 5.6. Let $B = B^{2,2}$, L the corresponding multiplicity array and $\lambda = (1, 0, 1, 2)$. Then

$$(\nu, J) = \quad \begin{array}{|c|}\hline \\ \hline \end{array}\ 0 \qquad \begin{array}{|c|c|l}\hline & & -1 \\ \cline{1-2} & \multicolumn{2}{|l}{-1} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \end{array}\ -1 \qquad \in \mathrm{RC}(L, \lambda)$$

corresponds to the tableau $b = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \in \mathcal{P}(B, \lambda)$. After step (1) of Definition 5.5 we have

$$(\nu', J') = \quad \begin{array}{|c|c|}\hline & \\ \hline \end{array}\ -1 \qquad \begin{array}{|c|c|l}\hline & & 1 \\ \cline{1-2} & \multicolumn{2}{|l}{0} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|l}\hline & & -1 \\ \cline{1-2} & \multicolumn{2}{|l}{-1} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline & \\ \hline \end{array}\ -1.$$

Then applying step (2) yields

$$\mathrm{pr}(\nu, J) = \quad \emptyset \qquad \begin{array}{|c|}\hline \\ \hline \end{array}\ 0 \qquad \begin{array}{|c|}\hline \\ \hline \end{array}\ -1$$

which corresponds to the tableau $\mathrm{pr}(b) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array}$.

LEMMA 5.7. [**33**, Lemma 4.10] *The map* pr *of Definition 5.5 is well-defined and satisfies* (5.2) *for* $1 \leq a \leq n-2$ *and* (5.3) *for* $0 \leq a \leq n-1$.

Lemma 7 of [**36**] states that for a single Kirillov–Reshetikhin crystal $B = B^{r,s}$ the promotion operator pr is uniquely determined by (5.2) for $1 \leq a \leq n-2$ and (5.3) for $0 \leq a \leq n-1$. Hence by Lemma 5.7 pr on $\mathrm{RC}(L)$ is indeed the correct promotion operator when L is the multiplicity array of $B = B^{r,s}$.

THEOREM 5.8. [**33**, Theorem 4.11] *Let L be the multiplicity array of $B = B^{r,s}$. Then* $\mathrm{pr} : \mathrm{RC}(L) \to \mathrm{RC}(L)$ *of Definition 5.5 is the promotion operator on rigged configurations.*

CONJECTURE 5.9. [**33**, Conjecture 4.12] *Theorem 5.8 is true for any $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$.*

Unfortunately, the characterization [**36**, Lemma 7] does not suffice to define pr uniquely on tensor products $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$. In [**8**] a bijection $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ is defined via a direct algorithm. It is expected that Conjecture 5.9 can be proven by showing that pr and Φ commute. Alternatively, an independent characterization of pr on tensor factors would give a new, more conceptual way of defining the bijection Φ between paths and (unrestricted) rigged configurations. A proof that the crystal operators f_a and e_a commute with Φ for $a = 1, 2, \ldots, n-1$ is given in [**8**].

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