Limit Theorems for Motions in a Flow with a Nonzero Drift.

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Abstract We establish diffusion and fractional Brownian motion approximations for motions in a Markovian Gaussian random field with a nonzero mean.

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1 Introduction.

The simplest model of a motion of a tracer particle in a turbulent medium, e.g. fluid, is given by the stochastic differential equation

$$d\boldsymbol{x}(t) = (\boldsymbol{v} + \varepsilon \mathbf{V}(t, \boldsymbol{x}))\, dt + \sqrt{2\kappa}\, d\mathbf{B}(t), \tag{1}$$

where the particle is located at $\mathbf{0}$ when $t = 0$. $\boldsymbol{v} \in R^d$ is a constant vector representing the mean velocity of the fluid, $\mathbf{V}(t, \boldsymbol{x}) = (V_1(t, \boldsymbol{x}), \cdots, V_d(t, \boldsymbol{x}))$ describes the *velocity fluctuations* and is assumed to be a zero mean stationary random field.

$\mathbf{B}(t)$ is a standard Brownian motion, whose presence accounts for the existence of the molecular diffusivity in the medium given by $\kappa \geq 0$. The parameter $\varepsilon > 0$ is supposed to indicate smallness of the fluctuation amplitude in comparison with the mean drift. Turbulent flows of that nature have been extensively studied in Physics of Fluids starting with the work of the British physicist G. Taylor in the 1920's (see [18] and also [6] for more references on the subject).

The question is to determine the long time behavior of the displacement $\boldsymbol{x}(t)$ from the statistics of the velocity and the size of the molecular diffusivity κ. The effect of molecular diffusivity is in many cases negligible if κ is small. We therefore set $\kappa = 0$ for simplicity of presentation. We assume also that the mean velocity $\boldsymbol{v} \neq 0$. The case of vanishing mean velocity is presented by the authors in a separate paper (see [4, 5]).

More specifically we set

$$\boldsymbol{y}_\varepsilon(t) := \boldsymbol{x}\left(\frac{t}{\varepsilon^2}\right) - \boldsymbol{v}\frac{t}{\varepsilon^2}. \tag{2}$$

We then have

$$\frac{d}{dt}\boldsymbol{y}_\varepsilon(t) = \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \boldsymbol{v}\frac{t}{\varepsilon^2} + \boldsymbol{y}_\varepsilon(t)\right).$$

A theorem describing the limiting behavior of the particle, proven in various forms in [10, 9, 12], says that $\boldsymbol{y}_\varepsilon(t)$, $t \geq 0$, considered as continuous trajectory stochastic processes tend weakly, as $\varepsilon \downarrow 0$, to a Brownian Motion whose covariance matrix is given by the so-called *Taylor - Kubo formula*

$$D^*_{ij} = \int_0^\infty \left\{\mathbf{E}[V_i(t, \boldsymbol{v}t)V_j(0, \mathbf{0})] + \mathbf{E}[V_i(t, \boldsymbol{v}t)V_j(0, \mathbf{0})]\right\} dt \tag{3}$$

(cf. [13]).

The version of the above theorem proven in [10] for $\boldsymbol{v} = 0$, requires, among others, the strong mixing assumption for the velocity field in the temporal variable. The key point of the argument is an approximation of the forward particle location valid for sufficiently large times that only takes into account the information available prior to a given instant, the so-called *frozen path approximation*. The temporal mixing property of the field guarantees then that the future particle displacement is almost independent of the past information thus, in consequence the diffusion approximation holds. This type of theorem can be also shown for a field with $\boldsymbol{v} \neq \mathbf{0}$ provided that its fluctuation is strongly mixing in space (see [9]). In this case one can project the spatial domain into time domain via the change of variables $\boldsymbol{x} := \boldsymbol{x} - \boldsymbol{v}t$, cf. (2), and then, using the fact that the mean drift size dominates, for small

ε, the amplitude of fluctuations it is possible to translate the spatial mixing property into temporal one for the velocity field considered in the new coordinates and, as a result, the argument of [10] can be applied.

In this paper we set out to give a description of the limiting behavior of scaled trajectories for a class of Gaussian fluctuation fields with *power-law spectrum.* This family of fields appears frequently in the mathematical theory of turbulence and is motivated by the seminal work of Kolmogorov [11], see also [2] for the shear layer flow case. In many interesting cases the fluctuation fields with that kind of spectrum do not satisfy the mixing condition either in time or space, thus the existing versions of limit theorems are not applicable.

In what follows we assume that $\mathbf{V}(t,\boldsymbol{x})$, $(t,\boldsymbol{x}) \in R \times R^d$, is a d-dimensional, time-stationary, space-homogeneous, time-Markovian, isotropic Gaussian velocity field defined over a probability space $(\Omega, \mathcal{V}, P)$ whose two-point covariance matrix $\mathbf{R} = [R_{ij}]$ is given by the Fourier transform:

$$R_{ij}(t,\boldsymbol{x}) = \mathbf{E}\left[V_i(t,\boldsymbol{x})V_j(0,\mathbf{0})\right] = \int\limits_{R^d} c_0(\mathbf{k}\cdot\boldsymbol{x})e^{-|\mathbf{k}|^{2\beta}t}\hat{R}_{i,j}(\mathbf{k})d\,\mathbf{k}. \tag{4}$$

Here and in the sequel we let $c_0(\phi) := \cos\phi$, $\;c_1(\phi) := \sin\phi$. $\mathbf{E}$ stands for the mathematical expectation calculated with respect to P. The spatial spectral density of the field $\hat{\mathbf{R}} = [\hat{R}_{i,j}]$ is given by

$$\hat{R}_{i,j}(\mathbf{k}) = \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}}\left(\delta_{i,j} - \frac{k_i k_j}{|\mathbf{k}|^2}\right), \tag{5}$$

where $a : [0,+\infty) \to R_+$, the so-called *ultraviolet cut-off*, is a certain compactly supported continuous function. The factor $\delta_{i,j} - k_i k_j/|\mathbf{k}|^2$ in (5) ensures that the velocity field is incompressible, i.e. $\nabla\cdot\mathbf{V}(t,\boldsymbol{x}) := \sum\partial_i V_i(t,\boldsymbol{x}) \equiv 0$. The function $\exp(-|\mathbf{k}|^{2\beta}t)$ in (4) is called the *time correlation function* of the velocity $\mathbf{V}$. The spectral density $\hat{\mathbf{R}}(\mathbf{k})$ is integrable over $\mathbf{k}$ for $\alpha < 1$. The ultraviolet cut-off is then needed to avoid divergence of the integral over $|\mathbf{k}|$. The parameter α is directly related to the decay exponent of $\mathbf{R}$. Namely, $\mathbf{R}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{\alpha-1}$ for $|\boldsymbol{x}| \gg 1$. As α increases to one, the spatial decay exponent of $\mathbf{R}$ decreases to zero and in consequence the strength of spatial correlation of modes increases. For $\alpha \geq 1$, the spectral density ceases to be integrable and needs an infrared cut-off, i.e. the origin should lie outside of the support of a. On the other hand, in order to apply the results of [9] or [10] one needs to assume that $\beta \leq 0$. Otherwise, i.e. when $\beta > 0$, the velocity field lacks the spectral gap and thus the strong mixing property (cf. [16]). In light of the foregoing discussion we therefore restrict our attention to the case $\alpha < 1$, $\beta > 0$, which corresponds to the velocity field having arbitrarily long scales but not the strong mixing property.

In addition, the particular form of the spectrum of the field allows us to assume without any loss of generality that it is jointly continuous in both t and $\boldsymbol{x}$ and is of C^∞ class in $\boldsymbol{x}$ a.s. (see e.g. [1]).

One can then pose the following question: what is the region in the (α,β) plane where the classical turbulent diffusion theorem, with the Taylor - Kubo formula (3), holds? It is easy to find the necessary condition by imposing the convergence of the integral appearing in (3). We have

$$D^*_{ij} = \int\limits_0^\infty R_{ij}(t,\boldsymbol{v}t)\;dt = \int\limits_{R^d}\left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right)\frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}}\int\limits_0^\infty \exp(-|\mathbf{k}|^{2\beta}t)c_0(k\boldsymbol{v}t)\;dt\;d\mathbf{k}, \tag{6}$$

where $k_{\boldsymbol{v}} := \mathbf{k}\cdot\boldsymbol{v}$. After a straightforward calculation of the integral over t one deduces that the rightmost part of (6) equals

$$\int\limits_{R^d} \frac{a(|\mathbf{k}|)|\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+d-2}(|\mathbf{k}|^{4\beta}+k_{\boldsymbol{v}}^2)}\left(\delta_{ij}-\frac{k_ik_j}{|\mathbf{k}|^2}\right)\,d\mathbf{k}. \tag{7}$$

Elementary calculations (cf. the computations made after (43) and (46) below) show that the finiteness of the expression (7) leads to the conditions

$$\alpha+\beta<1 \quad \text{when } \beta<1/2 \tag{8}$$

or

$$\alpha<1/2\le\beta<1. \tag{9}$$

It turns out that these conditions are also sufficient to claim that the particle trajectory approximates the Brownian motion in the sense of weak convergence of continuous path stochastic processes. This is essentially the contents of our Theorem 1 below.

One can also ask what happens in the case when conditions (8), (9) are violated. The fact that the integral in (6) diverges suggests "slower" than ε^{-2} scaling of the temporal variable in (2). We define

$$\boldsymbol{y}_\varepsilon(t) := \boldsymbol{x}\left(\frac{t}{\varepsilon^{2\delta}}\right)-\boldsymbol{v}\frac{t}{\varepsilon^{2\delta}} \tag{10}$$

for a certain suitably chosen $\delta\in(1/2,1)$. The limit of the stochastic processes given by the trajectories turns out then to be a fractional Brownian motion. This result is contained in Theorem 2.

It is interesting, in our judgement, to compare the results of the present article with those of [4] and [5]. It was proven there that the diffusive regime for the motion in the field without the drift, i.e. $\boldsymbol{v}=0$, comprises of (α,β) satisfying $\alpha+\beta<1$ (see [5]). When, on the other hand, $\alpha+\beta>1$, the properly scaled trajectories approximate a fractional Brownian motion (see [4]). As a result we see that inthe present setting the diffusivity regime is enlarged for $\beta<1/2$. We call this phenomenon, after [3], the *sweeping effect*. It can be explained intuitively as follows. For a given wavenumber of magnitude $|\mathbf{k}|$ one can see from (4) that the *relaxation time* needed for a significant temporal decorrelation of the fluctuation field is of order $\tau\sim|\mathbf{k}|^{-2\beta}$. On the other hand the *sweeping time* necessary for a significant decorrelation related to the mean drift convection is of order of magnitude $\tau_{\boldsymbol{v}}\sim|\mathbf{k}|^{-1}$. It is much shorter than the relaxation time when $\beta\ge1/2$. In that regime therefore the mean drift is responsible for the chaotic behavior of the velocity field and in consequence the diffusive approximation holds when $\alpha\le1/2$. On the other hand for $\beta<1/2$ the effect of the mean drift is negligible and the limiting particle motion is the result of a subtle balance between the spatial and temporal chaotic properties of the field as described in [4, 5].

To simplify the presentation we focus our attention only on the case when α,β satisfy in addition

$$\alpha+2\beta<2. \tag{11}$$

This condition can be proven inessential for the validity of our results. The more technically involved case when $\alpha+2\beta\ge2$ will be the subject of a subsequent article.

We summarize our results in the form of the following two theorems.

Theorem 1 *Suppose that*

1) $\boldsymbol{v} \neq 0$ *and* (11) *holds.*

2) $\mathbf{V}(t, \boldsymbol{x})$, $(t, \boldsymbol{x}) \in R \times R^d$, *is a zero mean, Gaussian velocity field whose correlation is given by* (4) *and* (5).

Then, for (α, β) *as specified by conditions* (8) *and* (9) *the scaled trajectories given by (2) converge weakly, when* $\varepsilon \downarrow 0$, *as continuous trajectory stochastic processes to a Brownian motion whose covariance matrix* $\mathbf{D}_0$ *is given by* (3).

Theorem 2 *Assume conditions* 1) *and* 2) *of Theorem* 1. *Then for* $1 > \alpha,\ \beta > 0$ *such that*

$$\alpha + \beta \geq 1 \quad \text{when } 1/2 \leq \alpha < 1 \tag{12}$$

one can find a unique $\delta_{\alpha,\beta} \in (1/2, 1)$ *and* $H = \frac{1}{2\delta_{\alpha,\beta}}$ *such that the scaled trajectories given by* (10) *converge weakly to a fractional Brownian Motion* $\mathbf{B}_H(t)$, $t \geq 0$, *i.e. the unique* H*-self-similar, Gaussian process with stationary increments (see* [17]*). We have*

$$\mathbf{E}\left[\mathbf{B}_H(t) \otimes \mathbf{B}_H(t)\right] = \mathbf{D}_{\alpha,\beta} t^{2H}, \tag{13}$$

with

$$\mathbf{D}_{\alpha,\beta} = \int\limits_{R^d} \Gamma_{\alpha,\beta} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right) \frac{a(0) d\mathbf{k}}{|\mathbf{k}|^{d-1}}. \tag{14}$$

where $\Gamma_{\alpha,\beta}$ *equals*

$$\frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} \qquad \text{when } \beta < 1/2,$$

$$\frac{1 - c_0(k_{\boldsymbol{v}})}{k_{\boldsymbol{v}}^2 |\mathbf{k}|^{2\alpha-1}} \qquad \text{when } \beta > \frac{1}{2},$$

$$\frac{(|\mathbf{k}|^2 - k_{\boldsymbol{v}}^2)\left(e^{-|\mathbf{k}|} c_0(k_{\boldsymbol{v}}) - 1 + |\mathbf{k}|\right) - 2|\mathbf{k}| k_{\boldsymbol{v}} \left(e^{-|\mathbf{k}|} c_1(k_{\boldsymbol{v}}) - k_{\boldsymbol{v}}\right])}{(|\mathbf{k}|^2 + k_{\boldsymbol{v}}^2)|\mathbf{k}|^{2\alpha-1}} \qquad \text{when } \beta = \frac{1}{2}.$$

In addition, $\delta_{\alpha,\beta}$ *equals*

$$\frac{\beta}{\alpha + 2\beta - 1} \qquad \text{when } \beta < 1/2, \tag{15}$$

$$\frac{1}{2\alpha} \qquad \text{when } \beta \geq 1/2. \tag{16}$$

2 Preliminaries.

The Spectral Theorem for real vector valued random fields (cf. e.g. [1]) implies that there exist two independent, identically distributed, real vector valued Gaussian spectral measures $\hat{\mathbf{V}}_l(t, \cdot) = (\hat{V}_{l,1}(t, \cdot), \cdots, \hat{V}_{l,d}(t, \cdot))$, $l = 0, 1$, such that

$$\mathbf{V}(t, \boldsymbol{x}) = \int \hat{\mathbf{V}}_0(t, \boldsymbol{x}, d\mathbf{k}), \tag{17}$$

where

$$\hat{\mathbf{V}}_0(t,\boldsymbol{x},d\mathbf{k}) := c_0(\mathbf{k}\cdot\boldsymbol{x})\hat{\mathbf{V}}_0(t,d\mathbf{k}) + c_1(\mathbf{k}\cdot\boldsymbol{x})\hat{\mathbf{V}}_1(t,d\mathbf{k}).$$

In the sequel (cf. e.g. Lemma 1) we also deal with the spectral measure given by

$$\hat{\mathbf{V}}_1(t,\boldsymbol{x},d\mathbf{k}) := -c_1(\mathbf{k}\cdot\boldsymbol{x})\hat{\mathbf{V}}_0(t,d\mathbf{k}) + c_0(\mathbf{k}\cdot\boldsymbol{x})\hat{\mathbf{V}}_1(t,d\mathbf{k}).$$

The field $\mathbf{V}$ is *Markovian* in the following sense. For any function $\psi \in \mathcal{S}(R^d,R)$ and $i \in \{1,\cdots,d\}$ we have

$$\mathbf{E}\left[\int \psi(\mathbf{k})\hat{V}_{l,i}(t,\boldsymbol{x},d\mathbf{k}) \mid \mathcal{V}_{-\infty,s}\right] = \int \mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}\psi(\mathbf{k})\hat{V}_{l,i}(s,\boldsymbol{x},d\mathbf{k}), \quad l=0,1. \tag{18}$$

Here, for any $a \le b$, $\mathcal{V}_{a,b}$ denotes the σ-algebra generated by $\mathbf{V}(t,\boldsymbol{x})$ with $(t,\boldsymbol{x}) \in [a,b]\times R^d$ and $\hat{V}_{l,i}(s,\boldsymbol{x},d\mathbf{k})$ is the relevant component of $\hat{\mathbf{V}}_l(s,\boldsymbol{x},d\mathbf{k})$. In what follows we also write $L^2_{a,b}$ for $L^2(\Omega,\mathcal{V}_{a,b},P)$.

We now introduce the key concepts appearing in the proofs of Theorems 1 and 2, namely χ_λ, the λ-*corrector* and $\mathbf{U}_\lambda$, the λ-*convector*. We set

$$\chi_\lambda(t,\boldsymbol{y}) := \int_t^{+\infty} \mathrm{e}^{-\lambda(s-t)}\mathbf{E}\left[\mathbf{V}(s,\boldsymbol{v}s+\boldsymbol{y}) \mid \mathcal{V}_{-\infty,t}\right] ds, \tag{19}$$

$$\mathbf{U}_\lambda(t,\boldsymbol{y}) := \mathbf{V}(t,\boldsymbol{v}t+\boldsymbol{y})\cdot\nabla\chi_\lambda(t,\boldsymbol{y}). \tag{20}$$

The differentiation used in (20) is understood in the mean square sense.

The procedure defining χ_λ and $\mathbf{U}_\lambda$ requires some justification at least to guarantee smoothness of the corrector field appearing on the right hand side of (20). This technical point is explained by the following.

Lemma 1 *For arbitrary $\lambda > 0$ the fields χ_λ, $\mathbf{U}_\lambda$ defined above are L^p integrable for any $p \ge 1$. L^p norms of those fields have the following hypercontractivity property:*

$$\left(\mathbf{E}|\chi_\lambda(t,\boldsymbol{y})|^p\right)^{1/p} \le C\left(\mathbf{E}|\chi_\lambda(t,\boldsymbol{y})|^2\right)^{1/2}, \tag{21}$$

$$\left(\mathbf{E}|\mathbf{U}_\lambda(t,\boldsymbol{y})|^p\right)^{1/p} \le C\left(\mathbf{E}|\mathbf{U}_\lambda(t,\boldsymbol{y})|^2\right)^{1/2}, \tag{22}$$

where the constant $C > 0$ does not depend on λ.

One can also select modifications of χ_λ and $\mathbf{U}_\lambda$ which are jointly stationary in the strict sense, continuous in both $(t,\boldsymbol{x})$ and C^∞ in $\boldsymbol{x}$.

The spectral representation of χ_λ is given by

$$\chi_\lambda(t,\boldsymbol{y}) = \chi_\lambda^{(0)}(t,\boldsymbol{y}) + \chi_\lambda^{(1)}(t,\boldsymbol{y})$$

with

$$\chi_\lambda^{(i)}(t,\boldsymbol{y}) := \int C_i(\mathbf{k},\lambda)\hat{\mathbf{V}}_i(t,\boldsymbol{v}t+y,d\mathbf{k}) \;\; i=1,2 \tag{23}$$

when

$$C_1(\mathbf{k},\lambda) := \frac{|\mathbf{k}|^{2\beta}+\lambda}{(|\mathbf{k}|^{2\beta}+\lambda)^2 + k_{\boldsymbol{v}}^2} \tag{24}$$

and

$$C_2(\mathbf{k},\lambda) := \frac{k_{\boldsymbol{v}}}{(|\mathbf{k}|^{2\beta}+\lambda)^2 + k_{\boldsymbol{v}}^2} \tag{25}$$

The proof of the lemma is fairly standard. We sketch here only its main points referring the reader interested in details to the relevant literature. L^p integrability and inequalities (21), (22) for the fields in question follow from the hypercontractivity property of L^p norms in Gaussian measure spaces related to the velocity field $\mathbf{V}$ (see e.g. Theorem 5.1 and its corollaries in [8]).

The existence of regular versions of the fields follows from the fact that there exists $h > 0$ such that for any integer $N > 0$ we can find a constant $C_{N,h} > 0$ depending on h, N only for which

$$\sum_{|\mathbf{m}|=N} \mathbf{E}|D^{\mathbf{m}}\mathbf{V}(t,\boldsymbol{x}) - D^{\mathbf{m}}\mathbf{V}(s,\boldsymbol{y})|^2 \le C_{N,h}\left[|t-s|^h + |\boldsymbol{x}-\boldsymbol{y}|^2\right]$$

for all $(t,\boldsymbol{x}),(s,\boldsymbol{y}) \in R \times R^d$. Here for any integral multiindex $\mathbf{m} = (m_1, \cdots, m_d)$ we define $D^{\mathbf{m}} := \partial_{x_1}^{m_1} \cdots \partial_{x_d}^{m_d}$. We can find the modifications of the fields with required degree of regularity using the aforementioned Theorem 5.1 of ibid in conjunction with Kolmogorov's classical result on the existence of continuous trajectory modification of a stochastic process (the random field version of that criterion can be found in e.g. [1] as the Corollary to Theorem 3.2.5).

Joint stationarity of the fields in question can easily be verified by an application of the results of [15].

3 Proof of Theorem 1.

We define a *scaled corrector along path* by

$$\chi_\varepsilon(t) := \varepsilon\chi_{\varepsilon^2}(\frac{t}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(t)) \tag{26}$$

and a *scaled convector along path* by

$$\mathbf{U}_\varepsilon(t) := \mathbf{U}_{\varepsilon^2}(\frac{t}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(t)). \tag{27}$$

Thanks to the divergence free structure of the velocity field $\mathbf{V}$ the processes $\chi_\varepsilon(t)$, $t \ge 0$, and $\mathbf{U}_\varepsilon(t)$, $t \ge 0$, are jointly strictly stationary (see Theorem 2, p. 500, of [15]).

The importance of the concepts of a corrector and convector is highlighted by the following lemma.

Lemma 2 *We have*

$$\boldsymbol{y}_\varepsilon(t) = -\chi_\varepsilon(t) + \chi_\varepsilon(0) + \int_0^t \chi_\varepsilon(s)\ ds + \int_0^t \mathbf{U}_\varepsilon(s)\ ds + M_\varepsilon(t). \tag{28}$$

Here $M_\varepsilon(t)$, $t \ge 0$, is a Brownian motion whose covariance matrix is given by $\mathbf{D}_\varepsilon$ where

$$\mathbf{D}_\varepsilon = \int_{R^d} \frac{|\mathbf{k}|^{2\beta} a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}\left[(|\mathbf{k}|^{2\beta}+\varepsilon^2)^2 + k_{\boldsymbol{v}}^2\right]}\left(\mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right) d\mathbf{k}. \tag{29}$$

In addition, there exists $\gamma > 0$ such that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\gamma}\mathbf{E}|\chi_\varepsilon(t)|^2 = 0 \qquad and \qquad \lim_{\varepsilon\downarrow 0}\varepsilon^{-\gamma}\mathbf{E}|\mathbf{U}_\varepsilon(t)|^2 = 0. \tag{30}$$

Proof. First we show (28). We start with the calculation of the so-called *pseudogenerator* of the path corrector, i.e. the process given by (cf. Chapter 3 of [14])

$$L\chi_\varepsilon(t) := \lim_{\delta\downarrow 0} \frac{\mathbf{E}\left[\chi_\varepsilon(t+\delta) \mid \mathcal{V}_{-\infty,t/\varepsilon^2}\right] - \chi_\varepsilon(t)}{\delta}, \tag{31}$$

where the limit is understood in the L^1 sense.

It is easy to observe via Taylor's expansion of $\chi_\varepsilon(\frac{t+\delta}{\varepsilon^2}, \boldsymbol{y})$ about $\boldsymbol{y}_\varepsilon(t)$ that

$$\chi_\varepsilon(t+\delta) = \varepsilon\chi_{\varepsilon^2}\left(\frac{t+\delta}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(t)\right) + R_\delta(t), \tag{32}$$

where

$$R_\delta(t) = \mathbf{U}_{\varepsilon^2}\left(\frac{t}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(t)\right)\delta + o(\delta).$$

Here $o(\delta)$ is understood as a function with takes values in the space random vectors with L^p integrable components for an arbitrarily chosen $p \geq 1$ and such that $o(\delta)/\delta$ vanishes in the L^p sense as $\delta \downarrow 0$.

Conditioning the first term on the right hand side with respect to the σ-algebra $\mathcal{V}_{-\infty,t/\varepsilon^2}$ we deduce from (19) and (20) that

$$\varepsilon\mathbf{E}\left[\chi_{\varepsilon^2}\left(\frac{t+\delta}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(t)\right) \mid \mathcal{V}_{-\infty,t/\varepsilon^2}\right] = \varepsilon \int\limits_{t+\delta/\varepsilon^2}^{+\infty} \mathrm{e}^{-\lambda(s-(t+\delta)/\varepsilon^2)}\mathbf{E}\left[\mathbf{V}(s, \boldsymbol{v}s + \boldsymbol{y}_\varepsilon(t)) \mid \mathcal{V}_{-\infty,t/\varepsilon^2}\right]. \tag{33}$$

The last expression, after applying Taylor's expansion this time about t/ε^2, can be rewritten as

$$\chi_\varepsilon(t) + \delta\left[\chi_\varepsilon(t) - \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \boldsymbol{v}\frac{t}{\varepsilon^2} + \boldsymbol{y}_\varepsilon(t)\right)\right] + o(\delta). \tag{34}$$

In consequence,

$$L\chi_\varepsilon(t) = \chi_\varepsilon(t) - \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \boldsymbol{v}\frac{t}{\varepsilon^2} + \boldsymbol{y}_\varepsilon(t)\right) + \mathbf{U}_\varepsilon(t). \tag{35}$$

Using the results of Chapter 3 of [14] we obtain that

$$M_\varepsilon(t) = \chi_\varepsilon(t) - \int\limits_0^t L\chi_\varepsilon(s)\, ds \tag{36}$$

is a continuous trajectory vector valued martingale. We write $M_\varepsilon = (M_{\varepsilon,1}, \cdots, M_{\varepsilon,d})$; a similar convention will be used for other vector valued fields appearing with subscript ε.

We notice that thanks to the aforementioned results of [14] the joint quadratic variation of martingales $M_{\varepsilon,i}(t)$, $M_{\varepsilon,j}(t)$, $t \geq 0$, $i, j = 1, \cdots, d$, equals

$$\langle M_{\varepsilon,i}, M_{\varepsilon,j}\rangle_t = \int\limits_0^t L\left\{M_{\varepsilon,i}(s)M_{\varepsilon,j}(s)\right\} ds. \tag{37}$$

After elementary calculations we get

$$L\left\{M_{\varepsilon,i}(t)M_{\varepsilon,j}(t)\right\}=\lim_{\delta\downarrow 0}\frac{\mathbf{E}\left[M_{\varepsilon,i}(t+\delta)M_{\varepsilon,j}(t+\delta)\mid\mathcal{V}_{-\infty,t/\varepsilon^2}\right]-M_{\varepsilon,i}(t)M_{\varepsilon,j}(t)}{\delta}\tag{38}$$

$$=\lim_{\delta\downarrow 0}\frac{\mathbf{E}\left[\chi_{\varepsilon,i}(t+\delta)\chi_{\varepsilon,j}(t+\delta)\mid\mathcal{V}_{-\infty,t/\varepsilon^2}\right]-\chi_{\varepsilon,i}(t)\chi_{\varepsilon,j}(t)}{\delta}-\left[\chi_{\varepsilon,i}(t)L\chi_{\varepsilon,j}(t)+\chi_{\varepsilon,j}(t)L\chi_{\varepsilon,i}(t)\right].$$

Using expansion (32) to represent $\chi_\varepsilon(t+\delta)$ in (38) we obtain that the utmost right hand side of (38) equals

$$L_{\varepsilon,i,j}(t;\boldsymbol{y}_\varepsilon(t))+U_{\varepsilon,i}(t)\chi_{\varepsilon,j}(t)+\chi_{\varepsilon,i}(t)U_{\varepsilon,j}(t)-\chi_{\varepsilon,i}(t)L\chi_{\varepsilon,j}(t)-\chi_{\varepsilon,j}(t)L\chi_{\varepsilon,i}(t),\tag{39}$$

where

$$L_{\varepsilon,i,j}(t;\boldsymbol{y}):=\varepsilon^2\lim_{\delta\downarrow 0}\frac{\mathbf{E}\left[\chi_{\varepsilon,i}(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y})\chi_{\varepsilon,j}(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y})\mid\mathcal{V}_{-\infty,t/\varepsilon^2}\right]-\chi_{\varepsilon,i}(\frac{t}{\varepsilon^2},\boldsymbol{y})\chi_{\varepsilon,j}(\frac{t}{\varepsilon^2},\boldsymbol{y})}{\delta}.\tag{40}$$

The corrector fields appearing in (40) are zero mean Gaussian therefore

$$\mathbf{E}\left[\chi_{\varepsilon,i}\left(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y}\right)\chi_{\varepsilon,j}\left(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y}\right)\mid\mathcal{V}_{-\infty,t/\varepsilon^2}\right]=m_{i,j}-\mathbf{E}m_{i,j}+\mathbf{E}\left[\chi_{\varepsilon,i}(0)\chi_{\varepsilon,j}(0)\right]$$

where

$$m_{i,j}=\varepsilon^2\mathbf{E}\left[\chi_{\varepsilon,i}(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y})\mid\mathcal{V}_{-\infty,\frac{t}{\varepsilon^2}}\right]\mathbf{E}\left[\chi_{\varepsilon,j}\left(\frac{t+\delta}{\varepsilon^2},\boldsymbol{y}\right)\mid\mathcal{V}_{-\infty,t/\varepsilon^2}\right].$$

With the help of (33) and (34) we deduce that

$$L_{\varepsilon,i,j}(t;\boldsymbol{y}_\varepsilon(t))=B_{\varepsilon,i,j}(t)-\mathbf{E}B_{\varepsilon,i,j}(t)\tag{41}$$

where

$$B_{\varepsilon,i,j}(t):=\left[\chi_{\varepsilon,i}(t)-\frac{1}{\varepsilon}V_i\left(\frac{t}{\varepsilon^2},\boldsymbol{v}\frac{t}{\varepsilon^2}+\boldsymbol{y}_\varepsilon(t)\right)\right]\chi_{\varepsilon,j}(t)+\left[\chi_{\varepsilon,j}(t)-\frac{1}{\varepsilon}V_j\left(\frac{t}{\varepsilon^2},\boldsymbol{v}\frac{t}{\varepsilon^2}+\boldsymbol{y}_\varepsilon(t)\right)\right]\chi_{\varepsilon,i}(t).$$

Combining (41) with (39) we deduce, using (35), that

$$L\left\{M_{\varepsilon,i}(t)M_{\varepsilon,j}(t)\right\}=-\mathbf{E}B_{\varepsilon,i,j}(t)=-\mathbf{E}B_{\varepsilon,i,j}(0),\tag{42}$$

with the last equality following from stationarity of the relevant processes along the particle trajectory that holds due to the results of [15]. A direct computation, using the formula for the expectation of a product of Gaussian random variables, shows that the right hand side of (42) equals $D_{\varepsilon,i,j}$, where $\mathbf{D}_\varepsilon:=[D_{\varepsilon,i,j}]$ is given by (29). This concludes the proof of (28).

Now we prove formula (30). Without any loss of generality we assume that $\boldsymbol{v}=(1,0,\cdots,0)$.

We then have

$$\mathbf{E}\left|\chi_\varepsilon(t)\right|^2=\mathbf{E}\left|\chi_\varepsilon(0)\right|^2\le C\varepsilon^2\int\limits_{|\mathbf{k}|\le K}\frac{d\mathbf{k}}{\left[(\varepsilon^2+|\mathbf{k}|^{2\beta})^2+k_1^2\right]|\mathbf{k}|^{2\alpha+d-2}}.\tag{43}$$

If we represent $\mathbf{k}=(k_1,\mathbf{l})$, where $\mathbf{l}\in R^{d-1}$ the rightmost part of (43) can be rewritten as

$$C\varepsilon^2 \int\int_{\sqrt{k_1^2+|\mathbf{l}|^2}\le K} \frac{dk_1 d\mathbf{l}}{\{[\varepsilon^2+(k_1^2+|\mathbf{l}|^2)^\beta]^2+k_1^2\}\,(k_1^2+|\mathbf{l}|^2)^{\alpha+(d-2)/2}} \tag{44}$$

$$\le C' \int\int_{\sqrt{k_1^2+l^2}\le K} \frac{\varepsilon^2 l^{d-2} dk_1 dl}{\{[\varepsilon^2+(k_1^2+l^2)^\beta]^2+k_1^2\}\,(k_1^2+l^2)^{\alpha+(d-2)/2}}.$$

After changing variables in the last integral according to the rule $l=r\cos\phi$, $k_1=r\sin\phi$ we infer that the leftmost part of (43) can be estimated from above by

$$C\int_0^K\int_0^{\pi/2} \frac{\varepsilon^2 dr d\phi}{[(\varepsilon^2+r^{2\beta})^2+r^2\phi^2]r^{2\alpha-1}}.$$

Here C denotes a generic constant independent of ε.

After an elementary calculation we find that the last integral equals

$$\int_0^K \frac{\varepsilon^2}{r^{2\alpha}(\varepsilon^2+r^{2\beta})}\arctan\left(\frac{\pi r}{2(\varepsilon^2+r^{2\beta})}\right)dr.$$

For α,β as specified by conditions (8), (9) the Dominated Convergence Theorem implies that this expression vanishes as $\varepsilon\downarrow 0$.

To show (30) we first observe that

$$\mathbf{E}\left|\mathbf{U}_\lambda(0,\mathbf{0})\right|^2 \le C\mathbf{E}\left|\mathbf{V}(0,\mathbf{0})\right|^2 \mathbf{E}\left|\nabla\chi_\lambda(0,\mathbf{0})\right|^2. \tag{45}$$

The gradient of the corrector satisfies

$$\mathbf{E}\left|\nabla\chi_\lambda(0,\mathbf{0})\right|^2 \le \int_{|\mathbf{k}|\le K} \frac{|\mathbf{k}|^2}{[(\lambda+|\mathbf{k}|^{2\beta})^2+k_1^2]\,|\mathbf{k}|^{2\alpha+d-2}}\ d\mathbf{k}. \tag{46}$$

Estimating precisely in the same way as we did for (43) we infer that the right hand side of (46) is less than or equal to a constant times

$$\int_0^K \frac{1}{r^{2\alpha-2}(\lambda+r^{2\beta})}\arctan\left(\frac{\pi r}{2(\lambda+r^{2\beta})}\right)dr.$$

This expression remains bounded as $\lambda\downarrow 0$ if only $\alpha+2\beta<2$. Consequently,

$$\limsup_{\lambda\downarrow 0}\mathbf{E}\left|\mathbf{U}_\lambda(0,\mathbf{0})\right|^2 < +\infty. \tag{47}$$

(47) holds also for any p-th absolute moment of $\mathbf{U}_\lambda$ with $p>0$ thanks to Lemma 1.

We also have

$$\mathbf{E}\left|\int_0^t \mathbf{U}_{\varepsilon^2}\left(\frac{s}{\varepsilon^2},\boldsymbol{y}_\varepsilon(s)\right)ds\right|^2 = 2\int_0^t ds\int_0^s \mathbf{E}\left\{\mathbf{U}_{\varepsilon^2}\left(\frac{s}{\varepsilon^2},\boldsymbol{y}_\varepsilon(s)\right)\cdot\mathbf{U}_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2},\boldsymbol{y}_\varepsilon(s_1)\right)\right\}ds_1$$

$$= 2\int_0^t ds \int_0^s \mathbf{E}\left\{\mathbf{U}_{\varepsilon^2}\left(\frac{s}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_1)\right) \cdot \mathbf{U}_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_1)\right)\right\} ds_1 \tag{48}$$

$$+\frac{2}{\varepsilon}\int_0^t ds \int_0^s ds_1 \int_{s_1}^s \mathbf{E}\left\{\left(\mathbf{V}\left(\frac{s_2}{\varepsilon^2}, \boldsymbol{v}\frac{s_2}{\varepsilon^2} + \boldsymbol{y}_\varepsilon(s_2)\right) \cdot \nabla \mathbf{U}_{\varepsilon^2}\left(\frac{s}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_2)\right)\right) \cdot \mathbf{U}_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_1)\right)\right\} ds_2$$

The first term of (48) equals

$$2\int_0^t ds \int_0^s \mathbf{E}\left\{\mathbf{U}_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2}, \mathbf{0}\right) \cdot \mathbf{U}_{\varepsilon^2}(0, \mathbf{0})\right\} ds_1 \tag{49}$$

due to the stationarity of the path convector.

Thanks to the incompressibility of the velocity field we have $\mathbf{E}\mathbf{U}_\lambda(t, \boldsymbol{y}) = \mathbf{0}$. That and well known properties of conditional expectations of second degree polynomial-like functionals of a Gaussian field then imply that

$$\mathbf{U}_\lambda(t, \boldsymbol{y}; 0) = \mathbf{V}(t, \boldsymbol{v}t + \boldsymbol{y}; 0) \cdot \nabla \chi_\lambda(t, \boldsymbol{y}; 0). \tag{50}$$

Here $\mathbf{V}(\cdot, \cdot; 0)$, $\nabla\chi_\lambda(\cdot, \cdot; 0)$ denote the orthogonal projections of $\mathbf{V}(\cdot, \cdot)$ and $\nabla\chi_\lambda(\cdot, \cdot)$ onto $L^2_{-\infty,0}$.

In consequence, (49) equals

$$2\int_0^t ds \int_0^s \mathbf{E}\left\{\left(\mathbf{V}\left(\frac{s_1}{\varepsilon^2}, \boldsymbol{v}\frac{s_1}{\varepsilon^2}; 0\right) \cdot \nabla\chi_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2}, \mathbf{0}; 0\right)\right) \cdot \mathbf{U}_{\varepsilon^2}(0, \mathbf{0})\right\} ds_1. \tag{51}$$

Using the Cauchy Schwarz inequality we estimate (51) by

$$C_1(\mathbf{E}|\mathbf{U}_{\varepsilon^2}(0, \mathbf{0})|^4)^{1/4}(\mathbf{E}|\mathbf{V}(0, \mathbf{0})|^4)^{1/4}\left(\int_0^t ds \int_0^s \mathbf{E}\left|\nabla\chi_{\varepsilon^2}\left(\frac{s_1}{\varepsilon^2}, \mathbf{0}; 0\right)\right|^2 ds_1\right)^{1/2}. \tag{52}$$

The fourth moment of $\mathbf{U}_{\varepsilon^2}$ remains bounded as $\varepsilon \downarrow 0$, see (47). Therefore, the expression in (52) can be estimated by a constant times the last factor of (52). The latter, with the help of (18) and (46), equals

$$\left[\int_0^t \int_{|\mathbf{k}| \le K} \frac{\varepsilon^2[1 - \exp(-\frac{2|\mathbf{k}|^{2\beta}s}{\varepsilon^2})]}{|\mathbf{k}|^{2\beta}} \times \frac{|\mathbf{k}|^2}{|\mathbf{k}|^{2\alpha+d-d}\left[(\varepsilon^2 + |\mathbf{k}|^{2\beta})^2 + k_1^2\right]}\, d\mathbf{k}\, ds\right]^{1/2}. \tag{53}$$

The expression (53) tends to 0 by virtue of the same change of variables as the one used above and the Dominated Convergence Theorem.

Finally, we consider the second term of (48). After a simple change of variables it can be rewritten as

$$\frac{2}{\varepsilon}\int_0^t ds \int_0^s ds_1 \int_{s_1}^s \mathbf{E}\left\{\left(\mathbf{V}\left(\frac{s_1}{\varepsilon^2}, \boldsymbol{v}\frac{s_1}{\varepsilon^2} + \boldsymbol{y}_\varepsilon(s_1)\right) \cdot \nabla \mathbf{U}_{\varepsilon^2}\left(\frac{s_2}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_1); \frac{s_1}{\varepsilon^2}\right)\right) \cdot \mathbf{U}_{\varepsilon^2}(0, \mathbf{0})\right\} ds_2. \tag{54}$$

An analogous argument to the one we used to deal with (49) shows that (54) vanishes as $\varepsilon \downarrow 0$, provided that

$$\frac{1}{\varepsilon^2}\mathbf{E}\left|\int_{s_1}^{s} \nabla \mathbf{U}_{\varepsilon^2}\left(\frac{s_2}{\varepsilon^2}, \boldsymbol{y}_\varepsilon(s_1); \frac{s_1}{\varepsilon^2}\right) ds_2\right|^2 = \frac{1}{\varepsilon^2}\mathbf{E}\left|\int_0^{s-s_1} \nabla \mathbf{U}_{\varepsilon^2}\left(\frac{s_2}{\varepsilon^2}, \mathbf{0}; 0\right) ds_2\right|^2 \tag{55}$$

$$\leq \frac{2}{\varepsilon^2}\mathbf{E}\left|\int_0^{s-s_1} \nabla \mathbf{U}^{(1)}_{\varepsilon^2}\left(\frac{s_2}{\varepsilon^2}, \mathbf{0}; 0\right) ds_2\right|^2 + \frac{2}{\varepsilon^2}\mathbf{E}\left|\int_0^{s-s_1} \nabla \mathbf{U}^{(2)}_{\varepsilon^2}\left(\frac{s_2}{\varepsilon^2}, \mathbf{0}; 0\right) ds_2\right|^2$$

tends to zero as $\varepsilon \downarrow 0$. Here

$$\nabla \mathbf{U}^{(i)}_{\varepsilon^2}(\frac{s_2}{\varepsilon^2}, \mathbf{0}; 0) := \nabla\left[\mathbf{V}(\frac{s_2}{\varepsilon^2}, \boldsymbol{v}\frac{s_2}{\varepsilon^2}; 0) \cdot \nabla \chi^{(i)}_{\varepsilon^2}(\frac{s_2}{\varepsilon^2}, \mathbf{0}; 0)\right] \quad i = 1, 2. \tag{56}$$

Using spectral representations (23), (24), (25) and (17) we can write the p, q-th entry of the matrix defined by (56) as being equal to

$$(-1)^i \int \exp\left\{-\frac{|\mathbf{k}|^{2\beta} s_2}{\varepsilon^2}\right\} C_i(\mathbf{k}, \varepsilon^2)\mathbf{k}\hat{V}_{1-i,p}\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{k}\right) \cdot \int l_q \exp\left\{-\frac{|\mathbf{l}|^{2\beta} s_2}{\varepsilon^2}\right\} \hat{\mathbf{V}}_1\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{l}\right)$$

$$-\int \exp\left\{-\frac{|\mathbf{k}|^{2\beta} s_2}{\varepsilon^2}\right\} C_i(\mathbf{k}, \varepsilon^2)\mathbf{k}k_q\hat{V}_{i,p}\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{k}\right) \cdot \int \exp\left\{-\frac{|\mathbf{l}|^{2\beta} s_2}{\varepsilon^2}\right\} \hat{\mathbf{V}}_0\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{l}\right).$$

Hence, each term of the rightmost part of (55) equals

$$\frac{2}{\varepsilon^2}\int_0^{s-s_1} ds_2 \int_0^{s-s_1} ds_2' \int\int\int\int \exp\left\{-\frac{(|\mathbf{k}|^{2\beta} + |\mathbf{l}|^{2\beta})s_2}{\varepsilon^2}\right\} \exp\left\{-\frac{(|\mathbf{k}'|^{2\beta} + |\mathbf{l}'|^{2\beta})s_2}{\varepsilon^2}\right\}$$

$$\times C_i(\mathbf{k}, \varepsilon^2)C_i(\mathbf{k}', \varepsilon^2)W(d\mathbf{k}, d\mathbf{l}, d\mathbf{k}', d\mathbf{l}'; s_2, s_2')$$

with W the signed measure given by

$$\sum_{p,q=1}^{d} \mathbf{E}\left\{\left[(-1)^i k_q \hat{V}_{1-i,p}\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{k}\right)\mathbf{k} \cdot \hat{\mathbf{V}}_1\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{l}\right) - k_q\hat{V}_{i,p}\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{k}\right)\mathbf{k} \cdot \hat{\mathbf{V}}_0\left(0, \boldsymbol{v}\frac{s_2}{\varepsilon^2}, d\mathbf{l}\right)\right]\right.$$

$$\left.\times \left[(-1)^i k_q' \hat{V}_{1-i,p}\left(0, \boldsymbol{v}\frac{s_2'}{\varepsilon^2}, d\mathbf{k}'\right)\mathbf{k}' \cdot \hat{\mathbf{V}}_1\left(0, \boldsymbol{v}\frac{s_2'}{\varepsilon^2}, d\mathbf{l}'\right) - k_q'\hat{V}_{i,p}\left(0, \boldsymbol{v}\frac{s_2'}{\varepsilon^2}, d\mathbf{k}'\right)\mathbf{k}' \cdot \hat{\mathbf{V}}_0\left(0, \boldsymbol{v}\frac{s_2'}{\varepsilon^2}, d\mathbf{l}'\right)\right]\right\}.$$

The above expectation is to be calculated treating the spectral measures as formal Gaussian random variables and subsequently applying the rules of calculating the expectation of products of Gaussians with the following relations for the covariance

$$\mathbf{E}\left[\widehat{\mathbf{V}}_l(0, \boldsymbol{x}, d\mathbf{k}) \otimes \widehat{\mathbf{V}}_{l'}(0, \boldsymbol{x}', d\mathbf{k}')\right] = \delta_{l,l'} c_0(\mathbf{k} \cdot (\boldsymbol{x} - \boldsymbol{x}'))\hat{\mathbf{R}}(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}')d\mathbf{k}d\mathbf{k}'.$$

After a straightforward computation we obtain that

$$W(d\mathbf{k}, d\mathbf{l}, d\mathbf{k}', d\mathbf{l}'; s_2, s_2') = 2|\mathbf{k}|^2 c_0(k_1\frac{s_2 - s_2'}{\varepsilon^2})c_0(l_1\frac{s_2 - s_2'}{\varepsilon^2})\mathrm{tr}\hat{\mathbf{R}}(\mathbf{k})\mathbf{k}^T\hat{\mathbf{R}}(\mathbf{l})\mathbf{k}$$

$$\times \delta(\mathbf{k}-\mathbf{k}')\delta(\mathbf{l}-\mathbf{l}')d\mathbf{k}d\mathbf{l}d\mathbf{k}'d\mathbf{l}'.$$

Thus each term on the utmost left hand side of (55) can be bounded from above by

$$C \iint\limits_{|\mathbf{k}|,|\mathbf{l}|\leq K} C_i^2(\mathbf{k},\varepsilon^2)\frac{|\mathbf{k}|^4\varepsilon^2\left[1-\exp\left\{-\frac{(|\mathbf{k}|^{2\beta}+|\mathbf{l}|^{2\beta})(s-s_1)}{\varepsilon^2}\right\}\right]^2}{(|\mathbf{k}|^{2\beta}+|\mathbf{l}|^{2\beta})^2(|\mathbf{k}||\mathbf{l}|)^{2\alpha+d-2}}\, d\mathbf{k}\ d\mathbf{l} \tag{57}$$

with the constant C independent of ε. For $\alpha+2\beta<2$ (57) tends to zero, as $\varepsilon\downarrow 0$, by virtue of the Dominated Convergence Theorem and the proof of the Lemma 2 is finished□

The proof of Theorem 1. By virtue of (28) we write

$$\boldsymbol{y}_\varepsilon(t) = M_\varepsilon(t) + R_\varepsilon(t) \tag{58}$$

where the remainder term $R_\varepsilon(t)$ consists of a stationary, i.e. $-\chi_\varepsilon(t)$, and an additive part

$$R_{a,\varepsilon}(t) := \int\limits_0^t \chi_\varepsilon(s)ds + \int\limits_0^t \mathbf{U}_\varepsilon(s)ds.$$

According to Lemma 2, $M_\varepsilon(t)$, $t\geq 0$, is a Brownian motion whose covariance matrix, given by (29), converges to $\mathbf{D}_0$ as $\varepsilon\downarrow 0$. In consequence, $M_\varepsilon(t)$, $t\geq 0$, converges weakly, when $\varepsilon\downarrow 0$, as processes with continuous trajectories (cf. e.g. [7]).

In order to conclude the proof of Theorem 1 it remains to be seen that for arbitrary $T>0$, $\delta>0$ we have

$$\lim_{\varepsilon\downarrow 0} P[\sup_{0\leq t\leq T}|R_\varepsilon(t)|\geq\delta]=0.$$

The assertions of Lemma 2 imply that we only need to show

$$\lim_{\varepsilon\downarrow 0} P[\sup_{0\leq t\leq T}|\chi_\varepsilon(t)|\geq\delta]=0.$$

Since χ_ε is stationary we can write

$$P[\sup_{0\leq t\leq T}|\chi_\varepsilon(t)|\geq\delta] \leq \frac{T}{\varepsilon^2}P[\sup_{0\leq t\leq T\varepsilon^2}|\chi_\varepsilon(t)|\geq\delta]. \tag{59}$$

Using again (58) to represent χ_ε we can estimate the right hand side of (59) by the sum

$$\frac{T}{\varepsilon^2}P[\sup_{0\leq t\leq T\varepsilon^2}[|R_{a,\varepsilon}(t)|+|\boldsymbol{y}_\varepsilon(t)|]\geq\delta] + \frac{T}{\varepsilon^2}P[\sup_{0\leq t\leq T\varepsilon^2}|M_\varepsilon(t)|\geq\delta] + \frac{T}{\varepsilon^2}P[|\chi_\varepsilon(0)|\geq\delta]. \tag{60}$$

The last term in (60) can be estimated using Chebychev's inequality by

$$\frac{\mathbf{E}|\chi_\varepsilon(0)|^p T}{\varepsilon^2\delta^p}. \tag{61}$$

Choosing p so that $\gamma\cdot p>4$ with γ given by Lemma 2, we see that the expression (61) tends to 0 as $\varepsilon\downarrow 0$ by virtue of (30).

The second term of (60) vanishes as $\varepsilon \downarrow 0$ thanks to

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-2}\mathbf{E}|M_\varepsilon(\varepsilon^2 T)|^2 = 0$$

and an elementary martingale inequality

$$\frac{T}{\varepsilon^2} P[\sup_{0\le t\le T\varepsilon^2} |M_\varepsilon(t)|] \ge \delta] \le \frac{\mathbf{E}|M_\varepsilon(\varepsilon^2 T)|^2 T}{\varepsilon^2\delta^2}. \tag{62}$$

The first term of (60) can be estimated by

$$C\left\{\frac{\varepsilon^2 T}{\delta^2}\mathbf{E}|\chi_\varepsilon(0)|^2 + \frac{\varepsilon^2 T}{\delta^2}\mathbf{E}|\mathbf{U}_\varepsilon(0)|^2 + \frac{T}{\varepsilon^2}P\left[\int_0^T |\mathbf{V}(s, \boldsymbol{v}s + \boldsymbol{y}(s))|\,ds \ge \frac{\delta}{\varepsilon}\right]\right\}. \tag{63}$$

The first two expressions in (63) tend to 0 thanks to Lemma 2. The last term of (63) vanishes as $\varepsilon \downarrow 0$, provided that $p > 2$. This is due to the fact that

$$\limsup_{\varepsilon\downarrow 0} \mathbf{E}\left[\int_0^T |\mathbf{V}(s, \boldsymbol{v}s + \boldsymbol{y}(s))|\,ds\right]^p \le \mathbf{E}\,|\mathbf{V}(0, \mathbf{0})|^p\, T^p < +\infty.$$

This finishes the proof of Theorem 1 $\square$

4 Proof of Theorem 2.

Throughout this section we assume that the scaled trajectory $\boldsymbol{y}_\varepsilon$ is given by (10) with $\delta_{\alpha,\beta}$ given by (15), (16). To avoid cumbersome notation we suppress writing the subscript of δ.

Define the path corrector and convector by

$$\chi_\varepsilon(t) := \varepsilon\chi_{\varepsilon^{2\delta}}(\frac{t}{\varepsilon^{2\delta}}, \boldsymbol{y}_\varepsilon(t)),$$

and

$$\mathbf{U}_\varepsilon(t) := \mathbf{U}_{\varepsilon^{2\delta}}(\frac{t}{\varepsilon^{2\delta}}, \boldsymbol{y}_\varepsilon(t)).$$

The following lemma can be proven exactly the same way as Lemma 2.

Lemma 3 *The scaled trajectory of a particle satisfies*

$$\boldsymbol{y}_\varepsilon(t) = -\chi_\varepsilon(t) + \chi_\varepsilon(0) + \int_0^t \chi_\varepsilon(s)ds + \varepsilon^{2(1-\delta)}\int_0^t \mathbf{U}_\varepsilon(s)ds + \varepsilon^{1-\delta}M_\varepsilon(t), \tag{64}$$

where $M_\varepsilon(t)$ is exactly as in the statement of Lemma 2.

Remark. Calculations analogous to those made after (43) show that for $\beta \le 1/2$ the exponent δ is chosen to make $\mathbf{E}|\varepsilon^{1-\delta}M_\varepsilon(t)|^2$ of order $O(1)$. Indeed,

$$\mathbf{E}|\varepsilon^{1-\delta}M_\varepsilon(t)|^2 = \varepsilon^{2-2\delta}\mathrm{tr}\mathbf{D}_\varepsilon t \simeq \varepsilon^{2\left(1+\delta\frac{1-\alpha-2\beta}{\beta}\right)} \tag{65}$$

and

$$\mathbf{E}|\chi_\varepsilon(t)|^2 = \varepsilon^2 \mathbf{E}|\chi_{\varepsilon^{2\delta}}(0,\mathbf{0})|^2 \simeq \varepsilon^{2\left(1+\delta\frac{1-\alpha-2\beta}{\beta}\right)}. \tag{66}$$

In the case when $\beta > 1/2$ we can deduce by the same argument that the terms on the left hand sides of (65) and (66) vanish as $\varepsilon \downarrow 0$.

We note here that $\delta < 1$ for $\alpha + \beta > 1$ and $\beta > 0$. Thanks to the assumption that $\alpha + 2\beta < 2$ we have (cf. (46))

$$\varepsilon^{4-4\delta}\mathbf{E}|\nabla\chi_{\varepsilon^{2\delta}}(0,\mathbf{0})|^2 \sim \varepsilon^{4-4\delta} \downarrow 0 \qquad \text{as } \ \varepsilon \downarrow 0. \tag{67}$$

This together with (45) implies that the term on the right hand side of (64) involving the convector vanishes in probability as $\varepsilon \downarrow 0$. In addition, we deduce by (67) that

$$\chi_\varepsilon(t) = \varepsilon\chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}},\mathbf{0}\right) + \varepsilon^{2(1-\delta)}\int_0^t \nabla\chi_{\varepsilon^{2\delta}}\left(\frac{s}{\varepsilon^{2\delta}},\boldsymbol{y}_\varepsilon(s)\right)\cdot\mathbf{V}\left(\frac{s}{\varepsilon^{2\delta}},\boldsymbol{v}\frac{s}{\varepsilon^{2\delta}}+\boldsymbol{y}_\varepsilon(s)\right)ds \tag{68}$$

$$= \varepsilon\chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}},\mathbf{0}\right) + o_\varepsilon(t).$$

Here and in the sequel we use the generic notation $o_\varepsilon(t)$ for any stochastic process which satisfies $\lim_{\varepsilon\downarrow 0}\mathbf{E}\sup_{0\le t\le T}|o_\varepsilon(t)|^2 = 0$, for any $T > 0$.

With this, (64) becomes

$$\boldsymbol{y}_\varepsilon(t) = -\varepsilon\chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}},\mathbf{0}\right) + \varepsilon\chi_{\varepsilon^{2\delta}}(0,\mathbf{0}) + \varepsilon\int_0^t \chi_{\varepsilon^{2\delta}}\left(\frac{s}{\varepsilon^{2\delta}},\mathbf{0}\right)ds + \varepsilon^{1-\delta}M_\varepsilon(t) + o_\varepsilon(t). \tag{69}$$

Lemma 4 *We have the following expansion of the corrector:*

$$\varepsilon\chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}},\mathbf{0}\right) = \varepsilon\chi_{\varepsilon^{2\delta}}(0,\mathbf{0}) + \varepsilon\int_0^{t/\varepsilon^{2\delta}}\chi_{\varepsilon^{2\delta}}(s,\mathbf{0})ds - \varepsilon\int_0^{t/\varepsilon^{2\delta}}\mathbf{V}(s,\boldsymbol{v}s)ds + \varepsilon^{1-\delta}\tilde{M}_\varepsilon(t) \tag{70}$$

where $\tilde{M}_\varepsilon$ is a Brownian motion which satisfies

i) *the i,j-th entry of its covariance matrix is given by (7).*

ii) *there exists $\gamma > 0$ such that*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{2(1-\delta)-\gamma}\mathbf{E}\left|M_\varepsilon(t) - \tilde{M}_\varepsilon(t)\right|^2 = 0. \tag{71}$$

Proof. The proofs of (70) and part i) go along the lines of the proofs of the corresponding parts of Lemma 2 so we leave this argument out. Actually, the fact that the spatial argument is fixed significantly simplifies the proof in the present case.

We focus on the proof of part ii). We first calculate the pseudogenerator

$$\mathcal{L}(t) := \varepsilon^{2(1-\delta)}\mathcal{L}\left|M_\varepsilon(t) - \tilde{M}_\varepsilon(t)\right|^2.$$

In light of (71) and (64) it is clear that

$$\mathcal{L}(t) = \varepsilon^2 \mathcal{L}|\mathbf{Z}(t, \boldsymbol{y}_\varepsilon(t))|^2 - \varepsilon^2 \mathbf{Z}(t, \boldsymbol{y}_\varepsilon(t)) \cdot \mathcal{L}\mathbf{Z}(t, \boldsymbol{y}_\varepsilon(t)) \tag{72}$$

where

$$\mathbf{Z}(t, \boldsymbol{y}) := \chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}}, \boldsymbol{y}\right) - \chi_{\varepsilon^{2\delta}}\left(\frac{t}{\varepsilon^{2\delta}}, \mathbf{0}\right).$$

The first term on the right hand side of (72) can be computed as L^1 limit of

$$\varepsilon^2 \lim_{r \downarrow 0} \frac{\mathbf{E}\left\{|\mathbf{Z}(t+r, \boldsymbol{y}_\varepsilon(t))|^2 \mid \mathcal{V}_{-\infty, t/\varepsilon^{2\delta}}\right\} - |\mathbf{Z}(t, \boldsymbol{y}_\varepsilon(t))|^2}{r}.$$

A direct computation similar to that performed after (31) shows that

$$\mathcal{L}(t) = 2\varepsilon^{2(1-\delta)} \int\limits_{R^d} \frac{|\mathbf{k}|^{2\beta} a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}[(\varepsilon^{2\delta} + |\mathbf{k}|^{2\beta})^2 + k_1^2]} [1 - c_0(\mathbf{k} \cdot \boldsymbol{x}_\varepsilon(t))]\, d\mathbf{k}. \tag{73}$$

Here, recall, $\boldsymbol{x}_\varepsilon(t) = \boldsymbol{v}t + \boldsymbol{y}_\varepsilon(t)$. For $2\beta > 1$ computations analogous to those performed after (43) or (46) convince us that the right hand side of (73) vanishes as $\varepsilon \downarrow 0$.

To estimate the mathematical expectation of the right hand side of (73) when $2\beta \le 1$ we break it into two disjoint integrals: the first over $|\mathbf{k}| \le \varepsilon^{\delta\varrho/\beta}$, with $0 < \varrho < 1$, and the second over its complement.

The expectation of the second integral can be estimated by

$$2\varepsilon^{2(1-\delta)} \int\limits_{K \ge |\mathbf{k}| \ge \varepsilon^{\delta\varrho/\beta}} \frac{|\mathbf{k}|^{2\beta+2-2\alpha-d}}{[(\varepsilon^{2\delta} + |\mathbf{k}|^{2\beta})^2 + k_1^2]}\, d\mathbf{k}. \tag{74}$$

Employing again the procedure used to estimate the last part in (43) we infer that (74) is less than or equal to a constant times

$$\varepsilon^{2(1-\delta)} \int\limits_{\varepsilon^{\frac{\delta\varrho}{\beta}}}^{K} \frac{r^{2\beta+1}}{r^{2\alpha}(r^{2\beta} + \varepsilon^{2\delta})^2}\, dr.$$

With the change of variables $r' := \varepsilon^{\delta\beta} r$ we conclude that the above expression vanishes, as $\varepsilon \downarrow 0$, at the rate $O(\varepsilon^\gamma)$ for some $\gamma > 0$ since $0 < \varrho < 1$.

The expectation of the first integral can be estimated as follows:

$$\begin{aligned}
&2\varepsilon^{3-4\delta} \int\limits_0^t ds \int\limits_{|\mathbf{k}| \le \varepsilon^{\delta\varrho/\beta}} \frac{|\mathbf{k}|^{2\beta} a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}[(\varepsilon^{2\delta} + |\mathbf{k}|^{2\beta})^2 + k_1^2]} \mathbf{E}\left[-\sin(\mathbf{k} \cdot \boldsymbol{x}_\varepsilon(s)) \mathbf{k} \cdot \mathbf{V}\left(\frac{s}{\varepsilon^{2\delta}}, \boldsymbol{x}_\varepsilon(s)\right)\right]\, d\mathbf{k} \\
&\le C\varepsilon^{3-4\delta} \int\limits_{|\mathbf{k}| \le \varepsilon^{\delta\varrho/\beta}} \frac{|\mathbf{k}|^{2\beta+3-2\alpha-d}}{[(\varepsilon^{2\delta} + |\mathbf{k}|^{2\beta})^2 + k_1^2]} d\mathbf{k}.
\end{aligned} \tag{75}$$

Via the same type of argument as used to estimate the previous integral we find that (75) is bounded from above by

$$\varepsilon^{\frac{2}{\alpha+2\beta-1}} \int\limits_0^{\varepsilon^{\delta(\varrho-1)/\beta}} \frac{r^{2\beta+2}}{r^{2\alpha}(r^{2\beta} + 1)^2} dr.$$

The divergence of the integral in (75), if occurs, can be made arbitrarily slow by choosing ϱ sufficiently close to one. Thus the expression in (75) vanishes for $\alpha+2\beta\le 2$. This concludes the proof of Lemma 4$\Box$

As a consequence of Lemma 4 we obtain

$$\boldsymbol{x}_\varepsilon(t)=\boldsymbol{y}_\varepsilon(t)+o_\varepsilon(t), \tag{76}$$

where

$$\boldsymbol{y}_\varepsilon(t)=\varepsilon\int_0^{t/\varepsilon^{2\delta}}\mathbf{V}(s,\mathbf{0})ds \tag{77}$$

is a Gaussian process whose covariance matrix equals

$$\mathbf{E}\left[\boldsymbol{y}_\varepsilon(t)\otimes\boldsymbol{y}_\varepsilon(t)\right]=2\varepsilon^2\int\limits_0^{t/\varepsilon^{2\delta}}ds\int\limits_0^s\mathbf{R}(s',\boldsymbol{v}s')\,ds'. \tag{78}$$

An elementary calculation shows that the right hand side of (78) tends to $\mathbf{D}t$ as $\varepsilon\downarrow 0$ with $\mathbf{D}$ given by (14). Hence $\boldsymbol{y}_\varepsilon$ converges weakly as $\varepsilon\downarrow 0$ to a fractional Brownian motion $\mathbf{B}_H(t)$ whose covariance is given by (13).

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