KLR algebras

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April 21, 2012

arXiv: 0909.1810 Joint with Aaron Lauda

The symmetric group

$\mathcal{S}_m$

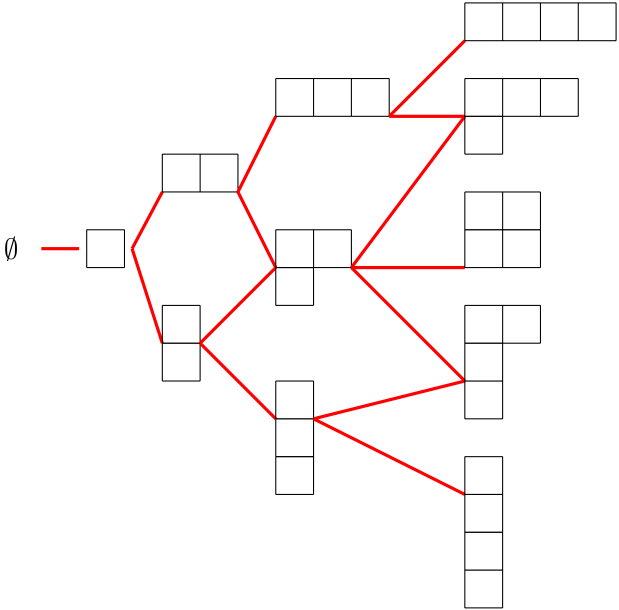
generators $\{s_1, s_2, \ldots, s_{m-1}\}$

relations $s_r^2 = 1$ quadratic

$s_r s_k = s_k s_r$ if $|r - k| > 1$ braid

$s_r s_k s_r = s_k s_r s_k$ if $r = k \pm 1$

Representation theory of the symmetric group



Branching Rule

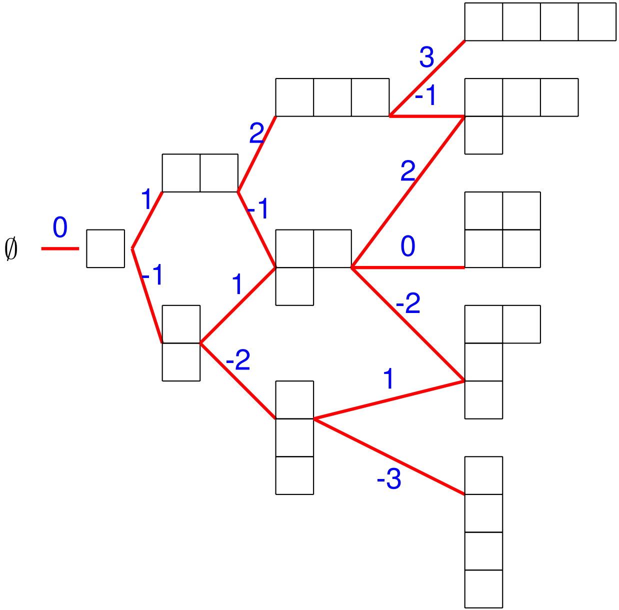
Refine restriction

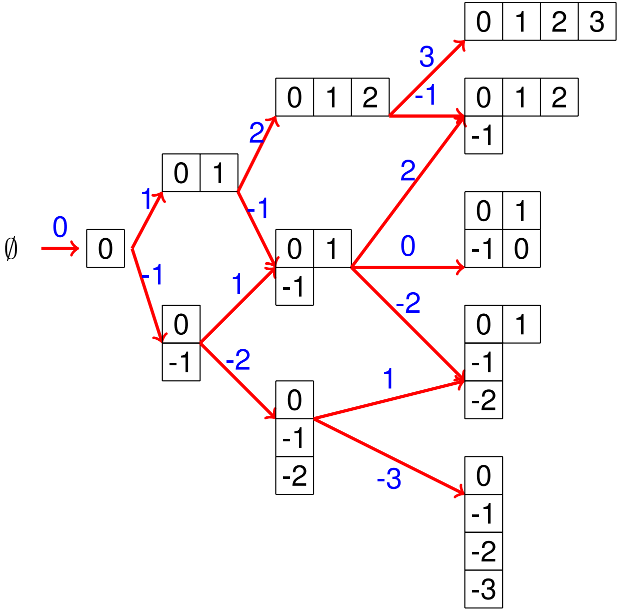
$$\mathcal{S}_{m-1} \subseteq \mathcal{S}_m$$

and more generally $\mathcal{S}_m \times \mathcal{S}_n \subseteq \mathcal{S}_{m+n}$.
Functors Res^m_{m-1} and Ind^m_{m-1}
Use the center $Z(\mathbb{Q}\mathcal{S}_m)$ to refine restriction/induction by projecting to blocks

$$\mathrm{Res}^m_{m-1} = \bigoplus_{i \in \mathbb{Z}} E_i \qquad \mathrm{Ind}^m_{m-1} = \bigoplus_{i \in \mathbb{Z}} F_i^{\Lambda_0}$$

label edges

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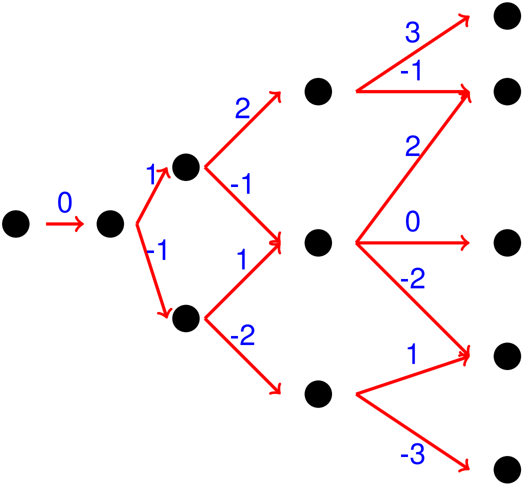
Crystal graph

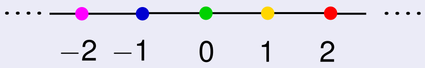


Lie algebra action

Functors E_i and $F_i^{\Lambda_0}$ yield an action of

$$\mathfrak{sl}_\infty \qquad \text{on} \qquad \bigoplus_{m\geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}\mathcal{S}_m\text{−mod})$$

$\mathfrak{sl}_\infty$ has Dynkin nodes $I = \mathbb{Z}$



yields integral highest weight module

$$V(\Lambda_0) \simeq \bigoplus_{m\geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}\mathcal{S}_m\text{−mod})$$

with crystal graph $B(\Lambda_0)$

Grothendieck ring

$K_0(\mathcal{C}\text{ategory}) =$

$$\{\text{ isom classes of objects } [A]\,\}/\langle\,[B] = [A] + [C] \text{ if } 0 \to A \to B \to C \to 0\,\rangle$$

$K_0 \iff$ projectives $\qquad G_0 \iff$ simples (finite dimensional)

For $K_0(\mathbb{Q}\mathcal{S}_m\mathrm{-mod})$ we can identify $[M]$ with its character χ

Beyond

$\mathcal{S}_m$ *categorifies* the $\mathfrak{g} = \mathfrak{sl}_\infty$ module $V(\Lambda_0)$

What about

- quantum groups $\mathcal{U}_q(\mathfrak{g})$?
- other $\mathfrak{g}$?
- other $V(\Lambda)$ for any $\Lambda \in P^+$?

q

Replace $\bigoplus_{m\geq 0} \mathbb{Q}\mathcal{S}_m$ with a *graded* ring R

$$q[M] = [M\{1\}]$$

in the Grothendieck ring, which is now a $\mathbb{Z}[q, q^{-1}]$-module with basis = isomorphism classes of simples

Start with

$$\mathfrak{g} = \mathfrak{sl}(2)$$

$$I = \{i\}$$

$R = \bigoplus_{m \geq 0} R(m\alpha_i)$

$R(m\alpha_i)$ has one simple (up to overall grading shift) called

$$L(i^m)$$

$R(0)$	$R(\alpha_i)$	$R(2\alpha_i)$	$R(m\alpha_i)$	$\cdots$

$$\emptyset \xrightarrow{i} L(i) \xrightarrow{i} L(i^2) \cdots \xrightarrow{i} L(i^m) \xrightarrow{i}$$

0	$-\alpha_i$	$-2\alpha_i$	$-m\alpha_i$

$$[E_i L(i^m)] = [\mathrm{Res}^{m\alpha_i}_{m\alpha_i - \alpha_i} L(i^m)] = [m]_i [L(i^{m-1})]$$

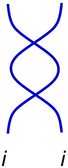
where $[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$ and $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

So $E_i^m [L(i^m)] = [m]_i! \emptyset$ and the divided powers $E_i^{(m)} = \frac{E_i^m}{[m]_i!}$ make sense

- $\bigoplus_{m \geq 0} G_0(R(m\alpha_i)-\mathrm{mod}) \simeq (\mathcal{U}_q^+)^*$ as $(\mathcal{U}_q^+)$-modules
- $\bigoplus_{m \geq 0} K_0(R(m\alpha_i)-\mathrm{pmod}) \simeq \mathcal{U}_q^-$ as bi-algebras (Khovanov-Lauda)
- $? \simeq V(\Lambda)$
  cyclotomic quotient

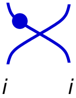
affine NilHecke algebra

$\partial_r^2 =$



$= 0$

$x_1\partial_1 - \partial_1 x_2 =$



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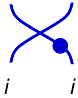

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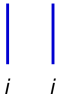
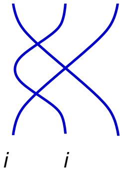


$\partial_r\partial_{r+1}\partial_r - \partial_{r+1}\partial_r\partial_{r+1} =$



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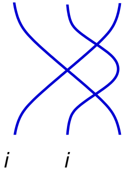
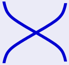


$= 0$

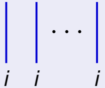
affine NilHecke algebra

deg () $= (\alpha_i, \alpha_i) = 2$

deg () $= -(\alpha_i, \alpha_i) = -2$

$R(m\alpha_i)$ is the affine NilHecke algebra (aka NH_m)
∂_r acts on polynomials f via

$$\partial_r(f) = \frac{f - s_r(f)}{x_r - x_{r+1}}$$

$R(m\alpha_i) \cong \mathrm{NH}_m$ with identity the idempotent $1_{i^m} =$ 

, where

$i^m := i \dots i$.

simples

Can realize $L(i^m)$ as $\mathbb{F}[x_1, \ldots, x_m]/\Lambda_m^+$ where Λ_m^+ is the ideal generated by symmetric polynomials with 0 constant term. (geometry $\to$ grading)

Can realize $L(i^m)$ as $\mathrm{Ind}_{(1,1,\ldots,1)}^m L(i) \boxtimes L(i) \boxtimes \cdots \boxtimes L(i)\{k\}$ (glue)

$$x_r^m L(i^m) = 0 \qquad x_r^{m-1} L(i^m) \neq 0$$

cyclotomic quotient

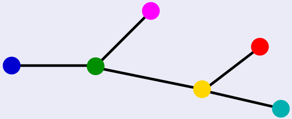
$$x_r^m L(i^m) = 0 \qquad x_r^{m-1} L(i^m) \neq 0$$

$R^\lambda(m\alpha_i) = R^n(m\alpha_i) = R(m\alpha_i)/\langle x_1^n \rangle$ is finite dimensional and will collapse if $m > n$

$$\begin{array}{ccccccccc} R^\lambda(0) & & R^\lambda(\alpha_i) & & R^\lambda(2\alpha_i) & & \cdots & & R^\lambda(n\alpha_i) \\ \emptyset & \xrightarrow{i} & L(i) & \xrightarrow{i} & L(i^2) & \xrightarrow{i} & \cdots & \xrightarrow{i} & L(i^n) \\ \lambda & & \lambda - \alpha_i & & \lambda - 2\alpha_i & & & & \lambda - n\alpha_i \\ n & & n-2 & & n-4 & & & & -n \end{array}$$

$\mathbf{U}_q^+$ for any Γ

Let Γ be an unoriented graph with set of vertices I.

$\Gamma =$ 

$\mathbf{U}_q^+$ is a $\mathbb{Q}(q)$-algebra; its integral form ${}_{\mathcal{A}}\mathbf{U}_q^+$ is the $\mathbb{Z}[q, q^{-1}]$-algebra with:

- generators: $E_i^{(n)} \quad i \in I$
- quantum Serre relations: $\sum_0^{a+1} (-1)^n E_i^{(a+1-n)} E_j E_i^{(n)} = 0 \quad$ if 

  where $a = a_{ij} = -\langle h_i, \alpha_j \rangle = -2\frac{i \cdot j}{i \cdot i}$

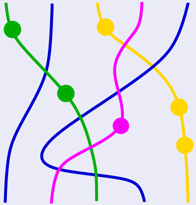
$\mathbf{U}_q^+$ is $Q^+ = \mathbb{N}[I]$ graded with $\deg(E_i) = \alpha_i$.

Definition of the algebra $R(\nu)$

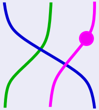
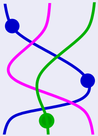
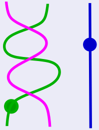
Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ.

Let $\nu = \sum_{i \in I} \nu_i \cdot \alpha_i$, for $\nu_i = 0, 1, 2, \ldots$

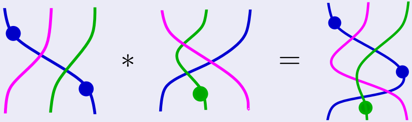
ν keeps track of how many strands of each color occur in a diagram

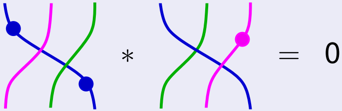


Form an abelian group by taking $\mathbb{Z}$-linear (or $\Bbbk$-linear) combinations of diagrams:

5  $-$ 2  $-$ 17 

Multiplication is given by stacking diagrams on top of each other when the colors match:

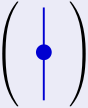


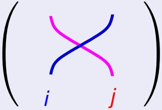


Definition

Given $\nu \in Q^+$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν, modulo planar braid-like isotopies and the following local relations:

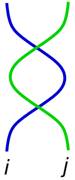
$R(\nu)$ local relations

deg  $= (\alpha_i, \alpha_i)$

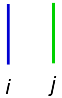
deg 

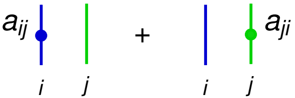
$= -(\alpha_i, \alpha_j)$

for $i \neq j$

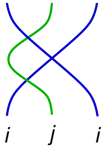


$=$

- 0 if $i = j$
- 

if $(\alpha_i, \alpha_j) = 0$
- 

if $(\alpha_i, \alpha_j) \neq 0$



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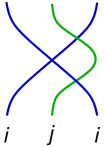
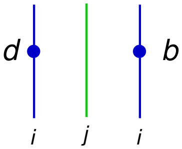
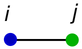
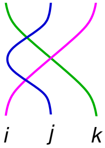


$$= \sum_{d+b=a_{ij}-1}$$



if 



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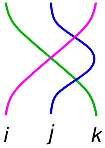


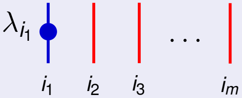
otherwise,

some of i, j, k may be equal

Cyclotomic quotients

For a given dominant integral weight $\lambda = \sum_{i \in I} \lambda_i \cdot \Lambda_i$ define the cyclotomic quotient $R^{\lambda}(\nu)$ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero $\longrightarrow$



$= \quad 0$

This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_m^{\lambda} := H_m / \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Cyclotomic quotients

The category of finitely-generated graded modules over the ring

$$R^\lambda = \bigoplus_{\nu \in Q^+} R^\lambda(\nu)$$

categorifies the integrable version of the representation V_λ of $\mathcal{U}_q(\mathfrak{g})$ of highest weight λ.

Theorem

1. (Webster, Kang-Kashiwara) As $\mathcal{U}_q(\mathfrak{g})$-modules $V(\lambda) \cong K_0(R^\lambda)$.
2. (Lauda-Vazirani) The simple R^λ-modules carry the structure of the corresponding crystal graph $B(\lambda)$.