

# Math 115a: Number Theory

## Final Exam

Please defend your answers in complete sentences. Since this is an exam, short justifications are good enough, but try to say things that hold water.

All problems are worth the same amount.

1. How many incongruent solutions  $(x, y)$  are there to the congruence equation

$$101x + 89y \equiv 3 \pmod{149}?$$

(You don't have to find the solutions, just count them. Note that all three of 89, 101, and 149 are prime numbers.)

2. How many of the prime residues of 67 have order 11? How many have order 13? (Note that 11, 13, and 67 are all prime.)
3. Prove that if a positive integer  $n$  is divisible by 30, then it is abundant (meaning that the sum of the proper divisors of  $n$  is more than  $n$ ).
4. Are there finitely many or infinitely many positive integers  $n$  such that  $\phi(n) = 96$ ? (As usual  $\phi(n)$  is the Euler  $\phi$  function, the number of incongruent prime residues modulo  $n$ .)
5. If  $p$  is a prime number, then a prime residue  $x$  is called a *quadratic residue* if it is the square of another residue. For example, the quadratic residues modulo 5 are 1 and 4. In general, how many non-congruent quadratic residues are there modulo  $p$ ?
6. Suppose that I'm thinking of an integer  $n$  which is congruent to 1 mod 2, 2 mod 3, 3 mod 5, 4 mod 7, 5 mod 11, and in general congruent to  $k$  modulo the  $k$ th prime  $p_k$  for all  $k$ . (In other words, it's an infinite list of conditions.) Prove that I'm not telling the truth; there is no positive or negative integer with all of these properties.
7. Let  $x_1, x_2, x_3,$  and  $x_4$  be the four incongruent solutions to the equation

$$x^2 \equiv 1 \pmod{10001}.$$

Find their product  $x_1x_2x_3x_4$  modulo 10001. (Note that the prime factorization is  $10001 = 73 \cdot 137$ .)