Final Exam 118A

Points: 220/

Name:
ID# :

1. **[140pts]** Consider the **diffusion equation** $u_t = u_{xx}$ for $0 < x < \pi, t > 0$ with mixed boundary conditions (bc),

$$u(0,t) = 0, \qquad u_x(\pi,t) = 0.$$

   (a) **[20pts]** Separate variables, and write down the two differential equations including bc's.

   (b) **[20pts]** Interpreting the differential equation for the spatial function as an eigenvalue equation for the operator $A = -\frac{d^2}{dx^2}$ on $(0,\pi)$, and with the appropiate bc's, find all (strictly) positive eigenvalues and their eigenfunctions explicitly.

(c) **[10pts]** Discuss whether there is also a 0-eigenvalue.

(d) **[20pts]** Determine whether there are negative eigenvalues.

(e) **[5pts]** Plot the eigenfunction for the lowest eigenvalue. Make sure you meet the bc's.

(f) **[15pts]** Show that with these bc's the operator A is self-adjoint (= symmetric, hermitian).

(g) **[5pts]** Using the self-adjointness of A, what can you say about possible complex eigenvalues?

(h) **[10pts]** Knowing now all eigenvalues of A, solve the time dependent equation from (1a).

(i) **[10pts]** Write down the general solution, $u(x,t)$.

(j) **[15pts]** Let $u(x,0) = \sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right)$ be the initial condition for $0 \leq x \leq \pi$. Compute the unique solution. (Think twice before you start a lengthy calculation.)

(k) **[10pts]** Calculate $u_\infty(x) = \lim_{t\to\infty} u(x,t)$.

2. **[60pts]** Let $f(x) = x^2$ with $0 \leq x \leq \pi$.

   (a) **[20pts]** Calculate the Fourier-cosine series of f. After you have calculated the coefficients, write down the series.

(b) **[10pts]** Let g_n be a sequence of functions defined on $[0, \pi]$, and g a function also defined on the same set $[0, \pi]$. Give a precise definition of what it means that *I*: g_n converges pointwise to g, and *II*: g_n converges in the L^2-sense to g.

(c) **[10pts]** Does the Fourier-series of f you calculated above converge pointwise (discuss the end-points separately), respectively in the L^2-sense to f?

(d) **[10pts]** Suppose, that this Fourier-series converges pointwise. Use this to derive the formula

$$\frac{\pi^2}{12} = \sum_{n=1,2,3,\ldots} (-1)^{n+1} \frac{1}{n^2}.$$

(e) **[10pts]** Knowing the Fourier-cosine series for $f(x) = x^2$, derive the Fourier-sine series for the function x.

3. **[20pts]** Derive the general formula for the diffusion equation with constant dissipation b,

$$u_t - ku_{xx} + bu = 0, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x).$$

Hint: Make the change of variables, $v(x,t) = u(x,t)e^{bt}$.