# The final.

**Instructions:**

1. Read these instructions through carefully. Ask me if you do not understand the instructions.
2. You have two hours to finish these problems.
3. Pick four out of the below six problems. If I find that there are more than four problems on the script I am grading, then I will grade the first four I find. If you do not want something to be graded, cross it out.
4. Each problem is worth 25 points.
5. If you are not sure what the problem wants, you can ask me. I will be happy to clarify questions, but, of course, I cannot clarify what the answer is.
6. The homework rules apply to this midterm:
   (a) Include you first name and your last name in the upper right hand corner of the first page.
   (b) A correct answer without steps logically leading to it earns 0 points.
   (c) Any solution that is too illegible or unclear to understand will not get full credit.
7. If you use a theorem to justify something, make sure you state what the theorem says. You must also explicitly check that the hypotheses of the theorem are satisfied.
8. All the problems are stolen and have been checked by both me and the grader, so there it is unlikely that they are wrong.

**Score:**

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| Total | |

**Problem 1: First order equations.** This problem is stolen from [2]. Consider the first order differential equation

$$x^2(\partial_x \varphi)(x,y) - y^2(\partial_y \varphi)(x,y) = 0 \tag{0.1}$$

with boundary condition 'at infinity' $\varphi(x,y) \to e^{\frac{1}{x}}$ as $y \to \infty$.

1. (a) The equation (0.1) can be written

   $$F(x,y) \cdot \nabla\varphi(x,y) = 0 \tag{0.2}$$

   where $F(x,y) = (x^2, -y^2)$. Sketch the vector-field $F$. (This diagram does not need to be too accurate, just as long as it contains the relevant information.)

   (b) Solve (0.1) using a change of variables.

   (c) Explain where your change of variables breaks down and how this is consistent with your diagram of $F$.

2. Now use variables separation (that is, write $\varphi(x,y) = X(x)Y(y)$ in (0.1)) to find a solution to (0.1) with the prescribed boundary conditions.

**Problem 2: Fourier series, I.** This problem is from [1].

1. The functions $\{\sin(x), \sin(2x), \sin(3x)\ldots\}$ are pairwise mutually orthogonal in $L^2[0,\pi]$. Explain how this fact allows us to find the formula for the coefficients of the Fourier sine series on $[0,\pi]$.

2. Consider the ordinary differential equation

   $$\frac{d^2\varphi}{dx^2}(x) + \omega^2\varphi(x) = \sin(ix) \text{ on } [0,\pi], \tag{0.3}$$

   where $i$ is an integer and $\omega$ is a non-integer. Suppose that the solution $\varphi$ can be expressed as a Fourier sine series

   $$\varphi(x) = \sum_{j=1}^{\infty} a_j \sin(jx) \text{ on } [0,\pi].$$

   By differentiating the sine series for $\varphi$ and substituting it back into (0.3), find the coefficients $a_j$, and thereby the solution $\varphi$.

**Problem 3: Fourier series, II.** This problem is from [3].

1. Find the Fourier cosine series of the function $f(x) = x(2\pi - x)$ on $[0,\pi]$.

2. Using a theorem, show that this Fourier series converges to $f$ in $L^2[0,\pi]$.

3. Using this Fourier series, show that $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$.

**Problem 4: Elliptic equations.** This problem is from [2].

1. Let $\Omega \subseteq \mathbf{R}^3$. Consider the general Neumann problem

   $$\Delta\varphi = f \text{ in } \Omega \tag{0.4}$$

   $$\nabla\varphi \cdot N = g \text{ on } \partial\Omega \tag{0.5}$$

where $N = (N_x, N_y, N_z)$ is the outward unit normal to $\partial\Omega$. Using the integration by parts formula,

$$\int_\Omega a(x,y,z)\frac{\partial b}{\partial x}(x,y,z)dxdydz = \int_{\partial\Omega} a(x,y,z)b(x,y,z)N_x(x,y,z)dS \tag{0.6}$$

$$- \int_\Omega \frac{\partial a}{\partial x}(x,y,z)b(x,y,z)dxdydz, \tag{0.7}$$

or otherwise, show that if $\varphi$ satisfies (0.4) and (0.5), then we must have

$$\int_\Omega f(x,y,z)\,dxdydz = \int_{\partial\Omega} g(x,y,z)\,dS. \tag{0.8}$$

This is known as a compatability condition.

2. Consider the Neumann problem

$$\Delta\varphi(x,y) = 0 \text{ in } [0,\pi]\times[0,\pi] \tag{0.9}$$

with boundary conditions

$$\partial_x\varphi(0,y) = 0 \text{ and } \partial_x\varphi(\pi,y) = 2\cos(y) \text{ in } [0,\pi] \tag{0.10}$$

$$\partial_y\varphi(x,0) = 0 \text{ and } \partial_y\varphi(x,\pi) = 0 \text{ in } [0,\pi]. \tag{0.11}$$

   (a) Show that the compatability condition (0.8) is satisfied for this problem.
   (b) Using variables separation, find $\varphi$ satisfying (0.9), (0.10) and (0.11).

**Problem 5: The heat equation.** This problem is from [2]. Consider the problem

$$(\partial_t - \Delta)\varphi(t,x) = 2 \text{ on } [0,T]\times[0,1] \tag{0.12}$$

with boundary conditions

$$\varphi(t,0) = 0 \text{ and } \varphi(t,1) = 0 \text{ on } [0,T] \tag{0.13}$$

and the initial condition

$$\varphi(0,x) = 0 \text{ on } [0,1]. \tag{0.14}$$

1. Show that if $\varphi(t,x) = \psi(x) + \Phi(t,x)$ satisfies (0.12) – where $\Phi$ is an arbitrary function and *not* the fundamental solution – then (0.12) can be decomposed into the equation

$$-\psi''(x) = 2 \text{ on } [0,1] \tag{0.15}$$

and the equation

$$(\partial_t - \Delta)\Phi = 0 \text{ on } [0,T]\times[0,1]. \tag{0.16}$$

2. Show that the boundary condition (0.13) can be decomposed into the boundary condition

$$\psi(0) = 0 \text{ and } \psi(1) = 0 \tag{0.17}$$

and the boundary condition

$$\Phi(t,0) = 0 \text{ and } \Phi(t,1) = 0 \text{ on } [0,T]. \tag{0.18}$$

3. Find $\psi$.
4. Show that the initial condition (0.14) can be written

$$\Phi(0,x) = -\psi(x) \text{ on } [0,1] \tag{0.19}$$

where $\psi$ is now known.

5. Using separations of variables, find $\Phi$ satisfying (0.16), (0.18) and (0.19).
6. Write down $\varphi$.

**Problem 6: The wave equation.**

1. Derive the d'Alembert's formula

$$\varphi(t,x) = \frac{1}{2}[g_1(x-ct) + g_1(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_2(y)dy \tag{0.20}$$

for the solution of the wave equation

$$(\partial_t^2 - c^2\Delta)\varphi = 0 \text{ on } [0,T] \times \mathbf{R} \tag{0.21}$$

with initial data

$$\varphi(0,x) = g_1(x) \text{ and } (\partial_t\varphi)(0,x) = g_2(x) \text{ on } \mathbf{R}. \tag{0.22}$$

2. Sketch $\varphi$ at two different instances in time, where $\varphi$ is the solution to (0.21) and (0.22) when $g_2 = 0$ and

$$g_1(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \end{cases}$$

This diagram does not need to be too accurate as long as the relevant information is contained.

3. Explain how (0.20) is consistent with the principle of causality.

# References


[1] William E. Boyce and Richard C. DiPrima. *Elementary differential equations and boundary value problems.* John Wiley & Sons Inc., New York, 1965.

[2] Tyn Myint-U and Lokenath Debnath. *Linear partial differential equations for scientists and engineers.* Birkhäuser Boston Inc., Boston, MA, fourth edition, 2007.

[3] E. C. Zachmanoglou and Dale W. Thoe. *Introduction to partial differential equations with applications.* Dover Publications Inc., New York, second edition, 1986.