Final Exam 118B on 03/23/02

Total points: 75
Student name:
Student ID#:

This exam contains 5 problems and 8 pages

1. [**20pts**] Let D be a bounded, connected and open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $-\Delta$ the Dirichlet Laplacian on D.

   (a) [**5pts**] Give precise definitions of a Dirichlet Green's function for both cases.

(b) **[10pts]** Prove uniqueness of Green's functions. You may choose a two or three dimensional domain.

(c) **[5pts]** Let u satisfy $\Delta u = f$ with f a nice function. Express the solution in terms of the Green's function and f.

2. [**10pts**] Find the Dirichlet Green's function for the Laplacian on the quadrant $D = \{(x, y) : x < 0, y < 0\}$. Discuss all properties the Green's function has to satisfy.

3. [**10pts**] Consider $u_{tt} - c^2 \Delta u + m^2 u = 0$ with $u = u(x, y, z, t)$. Prove that

$$E = \frac{1}{2} \int \left(u_t^2 + c^2 (\nabla u)^2 + m^2 u^2 \right) \, dxdydz$$

is constant. Assume that E is well-defined, i.e. finite, and that u and ∇u decay fast enough to 0 at infinity. Provide reasons.

4. **[10pts]** Solve the three-dimensional wave equation, $u_{tt} - c^2 \Delta u = 0$ in $\mathbb{R}^3$ with initial data $u(x, y, z, 0) = x^2 + y^2 + z^2$, and $u_t(x, y, z, 0) = 0$. Eventually, check your solution.

5. **[25pts]** Consider the diffusion equation $u_t = k\Delta u$ $(k > 0)$ on a bounded, open domain, $D \subset \mathbb{R}^2$ with homogeneous *Neumann* boundary conditions, and initial condition $u(x, 0) = \phi(x)$.

   (a) **[5pts]** Show explicitly why one is interested in knowing the eigenvalues and eigenvectors of $-\Delta$ on D. What are the conditions at the boundary of D?

   (b) **[5pts]** What can you say about these eigenvalues and eigenvectors? Provide one proof of one of your statements.

(c) **[5pts]** Suppose we know all eigenvalues and eigenvectors of $-\Delta$ from the last part, write down the solution explicitly involving only the knowledge of the eigenvalues, its eigenvectors and the initial data ϕ.

(d) **[5pts]** Compute $\lim_{t\to\infty} u(x,t)$ for $x \in D$.

(e) **[5pts]** Suppose we had taken *Dirichlet* boundary conditions instead, compute $\lim_{t\to\infty} u(x,t)$ for $x \in D$.