MAT119A Final Exam
March 18, 2005

Write solutions in the exam booklet provided. Start each problem on a new page. Be sure to label diagrams carefully and justify all of your answers. Remember to print your name on the exam booklet. Good Luck!

1. Consider the equation

$$\frac{dx}{dt} = \alpha x - \frac{x}{1+x}, \tag{1}$$

with $\alpha > 0$ and $x > -1$.

(a) Calculate the location of all fixed points x^* of equation 1 as a function of the parameter α.

(b) Determine the stability of these fixed points <u>analytically</u>. Use graphical means to determine stability of fixed points when the analytical method is inconclusive.

(c) Plot the bifurcation diagram for the system using α as the control parameter. Indicate the stability of the fixed points on the diagram. At what values of x and α does a bifurcation occur? What type of bifurcation is it?

(d) What type of bifurcations would occur if

$$\frac{dx}{dt} = \alpha x - \frac{x}{1+x} + \beta, \quad \beta \neq 0.$$

<u>Briefly</u> explain why?

2. Consider the nonlinear system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= x(x - x^2 - y) \\ \frac{dy}{dt} &= y(x - 1) \end{aligned} \tag{2}$$

(a) Find all fixed points and use linear stabilty analysis to assess their stability. What conclusions can be made about the flow around the fixed point in the nonlinear system?

(b) Plot the nullclines of the system. Indicate the direction of the flow on the nullclines and in the different regions of phase space.

(c) On a new plot, re-sketch the nullclines and sketch trajectories in the phase plane.

(d) Calculate the index of the fixed point at (0,0). By examining trajectories in the phase plane, determine the stability of the fixed point at (0,0) for the nonlinear system? What type of fixed point is (0,0)?

3. Consider the equation

$$\frac{d^2x}{dt^2} = x - x^3. \tag{3}$$

(a) Find the energy function $E(x, x')$ for equation 3. Show that E is constant along trajectories (i.e. $\frac{dE}{dt} = 0$.)

(b) Plot level sets (contours) of $E(x, x')$.

(c) Sketch trajectories in the x, x'-phase plane. How do the trajectories relate to the level sets of $E(x, x')$? What is the asymptotic behavior of the system, as $t \to \infty$, of the trajectory starting at (2,1)?

(d) What type of fixed point is (1,0)? What is it's stablity? Give an argument why conservative systems cannot have attracting fixed points.

4. Consider the nonlinear system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= -x(x^2 + y^2) \\ \frac{dy}{dt} &= -y(x^2 + y^2) \end{aligned} \tag{4}$$

(a) Use linear stabilty analysis to assess the stability of the unique fixed point at (0,0). What conclusions can be made about the flow around the fixed point in the <u>nonlinear</u> system?

(b) Show that system 4 is a gradient system and find the potential function $V(x, y)$.

(c) Use $V(x, y)$ to show that (0,0) is a <u>global</u> attractor ?

(d) Show that trajectories system 4 are in the direction of the $-grad(V)$.

5. Consider the nonlinear system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= y - x(r^4 - \mu(r^2 - 1) - 1) \\ \frac{dy}{dt} &= -x - y(r^4 - \mu(r^2 - 1) - 1) \end{aligned} \tag{5}$$

where $r^2 = x^2 + y^2$.

(a) Rewrite the system in polar coordinates.

(b) For $\mu = 0$, show that the circle $r = 2$ is a trapping region, i.e. show $r' < 0$ on the circle $r = 2$. Use the Poincare-Bendixson, theorem to show that there is a periodic orbit in the the circle $r = 2$.

(c) Show that there is a Hopf Bifurcation at (0,0) and $\mu = 1$. Is the Hopf bifurcation supercritical or subcritical? (hint: examine the system in polar coordinates to determine this).