Math 127 A 2

Winter quarter, 2005

Final exam

You may quote and use results from the book or from lectures.

Please write your answers on the exam paper and include extra pages if there is not enough space.

All questions have the same number of marks.

Problem 1

Suppose that (M, d) and (P, d') are metric spaces and $X \subset M$ and $Y \subset P$ are subsets.

- If $M \times P$ is given the maximum metric

  $$D((x, y), (x', y')) = max\{d(x, x'), d'(y, y')\}$$

  show that a set of the form $C \times D$ is open in $M \times P$ if and only if it contains a product of balls of the form $M_\epsilon(x) \times P_\epsilon(y)$, for any point $(x, y) \in C \times D$. Deduce that $C \times D$ is open in $M \times P$ if and only if C is open in M and D open in P.

- Prove that the projection maps $\pi_1 : M \times P \to M$, $\pi_2 : M \times P \to P$ are open maps, i.e the images of an open set in $M \times P$ by p_1 or p_2 is open in M or in P respectively.

- Prove that if $X \times Y$ is connected, then both X and Y are connected.

Problem 2

Suppose that $f : M \to \mathbb{R}$ is a continuous function from a metric space M to the real numbers.

- Prove that $f^{-1}(r)$ is closed in M for any $r \in \mathbb{R}$.

- Deduce that if M is compact, then $f^{-1}(r)$ is compact for any $r \in \mathbb{R}$.

- Further deduce that if $M = [a, b] \subset \mathbb{R}$, then there is a smallest and largest number x for which $f(x) = r$, for any fixed value $r \in \mathbb{R}$.

Problem 3

Define an equivalence relation on a set X.

Which of the following relations are equivalence relations? Give brief justifications for your answers.

- $X = \mathbb{Z}$, the integers, with $x \sim y$ if $x^2 - y^2 = 3k$ for some integer $k \in \mathbb{Z}$.
- $X = \mathbb{Z}$ with $x \sim y$ if $x \leq y$.
- $X = \mathbb{Z}$ with $x \sim y$ if $2x = 3y$.
- $X = \mathbb{R}^2$ with $(x, y) \sim (x', y')$ if $x^2 + y^2 = x'^2 + y'^2$.

Problem 4

- Define a cut of rational numbers and explain the difference between a rational cut and an irrational cut.
- Show that $\{r + \bar{r} : r^2 < 2, \bar{r}^2 < 3, r \in \mathbb{Q}, \bar{r} \in \mathbb{Q}\} \cup \{r : r < 0, r \in \mathbb{Q}\}$ is a cut representing the real number $\sqrt{2} + \sqrt{3}$.
- Is $\{r\bar{r} : r^2 < 2, \bar{r} < \frac{1}{3}, r \in \mathbb{Q}, \bar{r} \in \mathbb{Q}\}$ a cut? Explain your answer.

Problem 5

- Show that the following is an inner product on $\mathbb{R}^2$;

$$< (x, y), (x', y') >= 2xx' + xy' + yx' + 2yy'.$$

- Explain why $d((x, y), (x', y')) = 2(x - x')^2 + 2(x - x')(y - y') + 2(y - y')^2$ gives a metric on $\mathbb{R}^2$.
- Explain why $\{(x, y) : 2x^2 + 2xy + 2y^2 \leq 1\}$ is a convex set in $\mathbb{R}^2$.

Problem 6

- Explain why the set S of quadratic numbers is countable, where $S = \{m + n\sqrt{d} : m, n, d \in \mathbb{Z}, d > 0\}$.
- Show that the subset $\bar{S} = \{r + \bar{r}\sqrt{2} : r, \bar{r} \in \mathbb{Q}\}$ of $\mathbb{R}$ is not complete.
- Are there any countable complete subsets of $\mathbb{R}$? Give a brief justification for your answer.

Problem 7

- Suppose that S is a compact subset of a metric space M. Explain why the boundary ∂S of S is compact.

- If S is a bounded subset of $\mathbb{R}^n$, explain why the closure $\bar{S}$ of S is compact.

- Find an example of a bounded subset S of a metric space M for which the closure $\bar{S}$ is not compact.

Problem 8

Define a metric on the space $\mathcal{F}$ of functions $f : [0,1] \to \mathbb{R}$ by $d(f,g) = max\{|f(x) - g(x)| : 0 \leq x \leq 1\}$.

- Show that $\mathcal{F}$ is complete.

- Prove that $S = \{f : 0 \leq f(x) \leq 1, f \in \mathcal{F}\}$ is closed and bounded.

- Construct a sequence in $S = \{f : 0 \leq f(x) \leq 1, f \in \mathcal{F}\}$ which has no convergent subsequence. (Hint: Note that the functions have no special properties, so for example $f(x_0) = 1, f(x) = 0$ for $x \neq x_0$ is a function in $\mathcal{F}$).

- Construct an open covering of $S = \{f : 0 \leq f(x) \leq 1, f \in \mathcal{F}\}$ which has no finite subcovering.