Math 127b: Advanced Calculus
Final Exam
(3/20/03)

Write your name and student ID number in the upper right-hand corner of this sheet and write your initials on each page of your exam.

Each problem is worth the same number of points. You **must** justify or defend your answer for each problem.

1. Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then

$$\sum_{n=1}^{\infty} a_n b_n$$

also does.

2. Choose the right word: If $f : [a,b] \to \mathbb{R}$ is integrable, then f is $\{$always,sometimes,never$\}$ the derivative of

$$F(x) = \int_a^x f(t)dt.$$

3. Let $f_n(x) = \cos nx$ on $\mathbb{R}$. Is each f_n uniformly continuous? Is the sequence (f_n) equicontinuous? Is it uniformly equicontinuous?

4. Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, *i.e.*, suppose that you know R but not the series. Suppose also that $0 < R < \infty$. Find the real numbers x for which the series must converge conditionally, the numbers x for which it might converge conditionally, and the numbers x for which it can't converge conditionally. (A series converges conditionally if it converges, but doesn't converge absolutely.)

5. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The mesh condition for integrability says that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $\text{mesh}(P) < \delta$, then

$$U(f,P) - L(f,P) < \varepsilon.$$

Let $f(x) = x^3$ on $[0,1]$. Since it is continuous, it is integrable. Taking $\varepsilon = 1/10$, find a corresponding δ.

6. Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ which is continuous at 0 but which is not integrable.

7. Prove that if $f : [a,b] \to \mathbb{R}$ is a continuous function, then $||f||_2 \le \sqrt{b-a}||f||_\infty$, in other words that

$$\sqrt{\int f^2} \le \sqrt{b-a} \max f.$$

8. The inner product

$$\langle f,g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

is not defined for all continuous functions on $\mathbb{R}$, but it is defined when

$$\int_{-\infty}^{\infty} f(x)^2 dx < \infty \qquad \int_{-\infty}^{\infty} g(x)^2 dx < \infty.$$

Give an example of two non-zero, continuous function on $\mathbb{R}$ with this property and that are orthogonal to each other.