Advanced Calculus: Final
Math 127B, Winter 2005

1. Define $f_n, g_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx^2}{1+n^2x^2}, \qquad g_n(x) = \frac{n^2x}{1+n^2x^2}.$$

Show that the sequences (f_n), (g_n) converge pointwise on $[0,1]$, and determine their pointwise limits. Determine (with proof) whether or not each sequence converges uniformly on $[0,1]$.

2. Find all points $x \in \mathbb{R}$ where the following power series converges:

$$\sum_{n=0}^{\infty} \frac{1}{1+n2^n} x^n.$$

3. (a) Prove that the following series converge on $\mathbb{R}$ to continuous functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \qquad g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

(b) Prove that g is differentiable on $\mathbb{R}$, and $g' = f$.

4. Let $a > 0$. Give a definition of the following improper Riemann integral as a limit of Riemann integrals:

$$\int_2^{\infty} \frac{1}{x(\log x)^a}\, dx.$$

For what values of a does this integral converge? (Hint. Use the substitution $u = \log x$.)

5. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is f Riemann integrable on $[0,1]$? Prove your answer.

6. Suppose that

$$F(x) = \begin{cases} -x^2 & \text{for } -1 \le x < 0, \\ x^2 + 2 & \text{for } 0 \le x \le 1. \end{cases}$$

Evaluate the Riemann-Stieltjes integral

$$\int_{-1}^{1} e^{x^2} dF(x).$$

Briefly justify your computations.

7. (a) Find the Taylor series of e^{-x} (at $x = 0$).

(b) Give an expression for the remainder $R_n(x)$ between e^{-x} and its Taylor polynomial of degree $n - 1$ involving an intermediate point y between 0 and x.

(c) Prove from your expression in (b) that the Taylor series for e^{-x} converges to e^{-x} for every $x \in \mathbb{R}$. (Don't use general theorems.)

8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 [\sin(1/x) - 2] & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Prove that $f(x)$ has a strict maximum at $x = 0$ (i.e. $f(0) > f(x)$ for all $x \neq 0$).

(b) Prove that f is differentiable on $\mathbb{R}$.

(c) Prove that f is not increasing on the interval $(-\epsilon, 0)$ and f is not decreasing on the interval $(0, \epsilon)$ for any $\epsilon > 0$.