December 9, 2008

Math 135A Final Exam, Fall 2008
Total Points = 120

Instructions: Work all problems in your Blue Book. This exam will not be collected.

Useful Information which you may assume as given:

1. For $|x| < 1$ we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \sum_{k=0}^{\infty} k^2x^k = \frac{x(1+x)}{(1-x)^3}, \sum_{k=0}^{\infty} k^3x^k = \frac{x(1+4x+x^2)}{(1-x)^4}$$

2. For all $z \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \mathrm{e}^z, \ \sum_{n=0}^{\infty} n\frac{z^n}{n!} = z\,\mathrm{e}^z, \ \sum_{n=0}^{\infty} n^2\frac{z^n}{n!} = (z+z^2)\,\mathrm{e}^z$$

,

$$\sum_{n=0}^{\infty} n^3\frac{z^n}{n!} = (z+3z^2+z^3)\,\mathrm{e}^z, \ \sum_{n=0}^{\infty} n^4\frac{z^n}{n!} = (z+7z^2+6z^3+z^4)\,\mathrm{e}^z$$

3. Recall: The density for a random variable with $N(0,\sigma^2)$ law is

$$p_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\ \mathrm{e}^{-x^2/(2\sigma^2)}.$$

   Here are some integrals involving p_σ:

$$\begin{aligned}
\int_0^\infty p_\sigma(x)\,dx &= \frac{1}{2} \\
\int_0^\infty x\,p_\sigma(x)\,dx &= \frac{\sigma}{\sqrt{2\pi}} \\
\int_0^\infty x^2\,p_\sigma(x)\,dx &= \frac{\sigma^2}{2} \\
\int_0^\infty x^3\,p_\sigma(x)\,dx &= \sqrt{\frac{2}{\pi}}\,\sigma^3 \\
\int_0^\infty x^4\,p_\sigma(x)\,dx &= \frac{3}{2}\,\sigma^4
\end{aligned}$$

.... **End of Useful Information** ....

Beginning of Exam Questions

1. Short Answers/Multiple Choice—Each question worth 5 points for a total of 25 points

   (a) Let X and Y be two discrete random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Give the definition that X and Y are *independent* random variables.

   (b) Let $\{X_n\}_{n\geq 1}$ denote a sequence of independent and identically distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume $\mu = \mathbb{E}(X_j) < \infty$ and $\sigma^2 = \text{var}(X_j) < \infty$, $\sigma^2 \neq 0$. State the *central limit theorem.*

   (c) Let X and Y be two discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

   $$\mathbb{E}\left(\mathbb{E}(Y|X)\right) = \mathbb{E}(Y)$$

   (d) Recall that X is said to be a *Poisson* random variable with parameter λ if

   $$\mathbb{P}(X = n) = \frac{\lambda^n}{n!}\, e^{-\lambda} \text{ for } n = 0, 1, 2, \ldots$$

   Find a closed form expressions for $\mathbb{E}(X)$ and $\text{var}(X)$ in terms of λ.

   (e) Suppose X is a normal (Gaussian) random variable with mean 0 and variance σ^2. Then the value of $\mathbb{E}(|x|^3)$ is

      i. 0
      ii. σ^2
      iii. $\sqrt{\frac{2}{\pi}}\,\sigma^3$
      iv. $2\sqrt{\frac{2}{\pi}}\,\sigma^3$
      v. The expected value does not exist, it is infinite.

2. The *Maxwell-Boltzmann distribution* for the speed v (=absolute value of the velocity) of a molecule in a dilute gas has density

   $$f_{MB}(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right), \quad 0 \leq v < \infty,$$

   where m is the mass of a gas molecule, T is the temperature and k_B is a constant called *Boltzmann's constant.*

   (a) (10 pts) In terms of the normal (Gaussian) density p_σ with mean 0 and variance σ^2 (see Useful Information), show that

   $$f_{MB}(v) = \frac{2}{\sigma^2} v^2\, p_\sigma(v) \text{ where } \sigma^2 = \frac{k_B T}{m}$$

(b) (10 pts) Show that the average speed of a gas molecule is (see Useful Information)

$$\mathbb{E}(v) = \left(\frac{8k_B T}{\pi m}\right)^{1/2}$$

(c) (10 pts) Show that the average kinetic energy of a gas molecule is $\frac{3}{2}\,k_B T$. That is, show (see Useful Information)

$$\mathbb{E}(\frac{1}{2}mv^2) = \frac{3}{2}\,k_B\,T$$

3. Recall that a random variable X is said to have *geometric distribution* with parameter q, $0 < q < 1$, if

$$\mathbb{P}(X = n) = q^{n-1}(1-q) \text{ for } n = 1, 2, \ldots$$

(a) 10 pts. In a Bernoulli trials process where $p = \mathbb{P}(\text{Success})$ and $q = \mathbb{P}(\text{Failure})$, $p + q = 1$, let X denote the time for the *first success*. Explain why

$$\mathbb{P}(X = n) = q^{n-1}p \text{ for } n = 1, 2, \ldots$$

(b) 5 pts. Let $G_X(s) := \mathbb{E}(s^X)$ where X is given in (a). Show that (assume $0 < s < 1$)

$$G_X(s) = \frac{ps}{1 - qs}$$

(c) 10 pts. In the *Coupon Problem* there are n different types of coupons and the collector wishes to collect all n coupons. At each trial a coupon is chosen at random. Each coupon is equally likely and the choices are independent. Let

$$T_n = \text{number of trials to collect all } n \text{ coupons}$$

Explain why

$$T_n = X_0 + X_1 + \cdots + X_{n-1}$$

where X_j is an independent geometric distributed random variable with

$$p_j = \frac{n-j}{n}, \quad j = 0, 1, \ldots, n-1.$$

(d) 10 pts. Find the generating function of T_n, i.e.

$$G_{T_n}(s) := \mathbb{E}(s^{T_n})$$

4. An unfair coin is flipped repeatedly where $\mathbb{P}(H) = p$, $\mathbb{P}(T) = q = 1 - p$. Let X be the number of flips until HT appears for the first time. Let $G_X(s) = \mathbb{E}(s^X)$ denote the generating function of X.

   (a) 25 pts. Show that

$$G_X(s) = \frac{pqs^2}{(1 - ps)(1 - qs)}$$

   (b) 5 pts. Show that

$$\mathbb{E}(X) = \frac{1}{pq}$$

   Hint: To start, condition on the outcome of the first flip.