MATH 21B
CALCULUS
FINAL EXAMINATION

SPRING QUARTER, 2002

Last Name: ______________________________

First Name: ______________________________

Student ID Number: ________________________

Section Number: _______

Instructions:

1. Write your name at the top of each page.
2. You may use only the following resources: a pencil, an eraser, a calculator, your brain.
3. Solve each problem on the paper provided, showing all your work *neatly.* Partial credit cannot be given for unintelligible responses. **You will receive no credit for unsupported answers.**
4. If you must, you may continue a solution on the back of any of the pages. If you do so, indicate clearly where the continuation can be found.
5. Express all numbers in exact arithmetic. Do not use decimal approximations. You need not numerically evaluate expressions involving square roots, logarithms, fractional powers, or constants such as π or e. That is, an answer in a form like $\sqrt{3} + \frac{\pi}{2} + \ln 2$ is acceptable. However, try to simplify your answers as much as possible.
6. Read and sign the following honor code pledge.

On my honor, I have neither given nor received any aid on this examination.

X ______________________________

Date: June 12, 2002.

Part I. (30 POINTS)

Compute the following

1. $\displaystyle\int \frac{\ln\ln x}{x\ln x}\,dx$

2. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{4-x^2}}$

3. $\displaystyle\int \frac{x-1}{x^2-5x+6}\,dx$

Part II. (20 POINTS)

1. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{e^{x/2}\,dx}{1+e^x}$.

2. Does the improper integral $\displaystyle\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx$ converge or diverge? Explain your reasoning.

Part III. (20 POINTS)

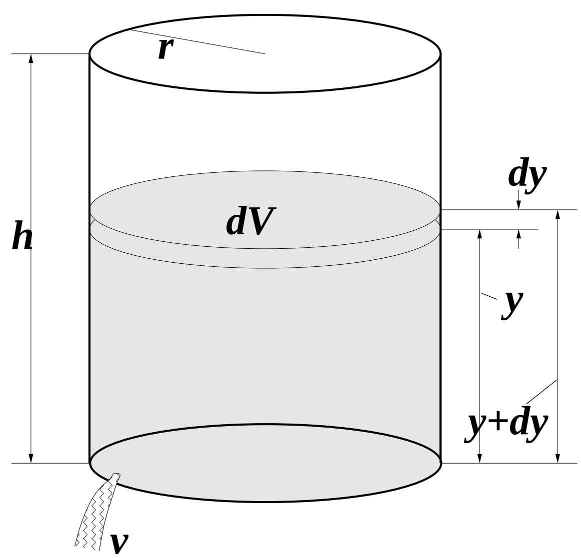
A cylindrical tank of radius r centimeters and height h centimeters is initially full of water. Water flows out from a small hole in the bottom of the tank at the rate $v = k\sqrt{y}$ cm^3/sec, where y is the height of water and k is a constant which depends on the diameter of the hole. Follow the steps below to find how much time will it take for the tank to empty.

1. Consider the "slice" of water from height y to $y + dy$. Find the volume, dV, of the "slice".

2. Assuming that dy is small, estimate the time, dt, necessary for the level to drop from $y + dy$ to y.

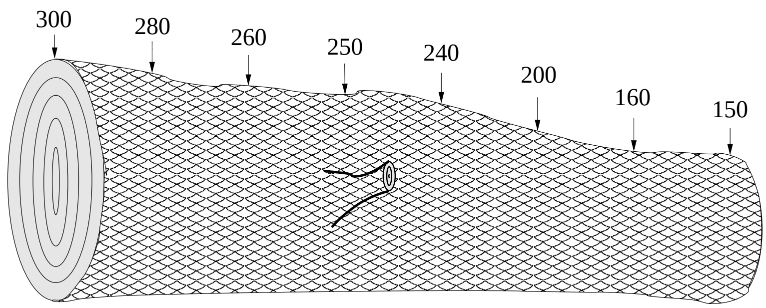
3. Set up an integral for evaluating the total time necessary for the tank to drain.

4. Evaluate the integral you obtained in #3.



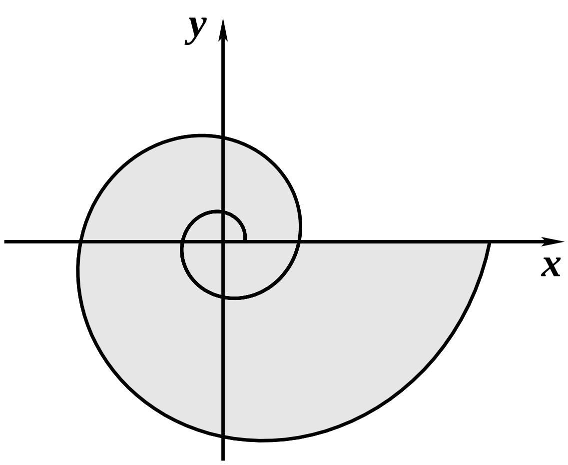
Part IV. (20 POINTS)

The diameter of a log, 7 meters long, is measured at 1 meter intervals. The measurements, in centimeters, are shown on the figure below. Using these measurements and the trapezoid rule, write down an expression for the approximate volume of wood contained in the log, in cubic centimeters.



Part V. (20 POINTS)

The equation $r = e^\theta$ describes a curve in polar coordinates. This curve is called a *logarithmic spiral* (see figure).



1. Compute the arc length (path length) of the spiral for $0 \le \theta \le 4\pi$.

2. Compute the area enclosed by the logarithmic spiral $r = e^\theta$ and the line segment $y = 0$, $e^{2\pi} \le x \le e^{4\pi}$. This is the area shaded in the figure.

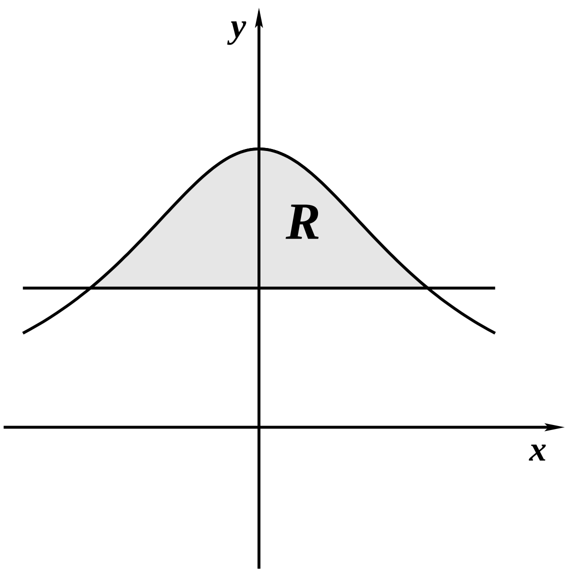
Part VI. (10 POINTS)

Let the function F be defined by $F(x) = \int_1^x \ln t \cos t \, dt$.

Find an expression for $\int_1^x \frac{\sin t}{t} \, dt$ in terms of $F(x)$. *Hint: use integration by parts.*

Part VII. (20 POINTS)

Consider the plane region R whose rectangular coordinates (x, y) satisfy the inequalities $\frac{1}{2} \leq y \leq \frac{1}{1+x^2}$ (see the figure).



1. A solid of revolution is made from this region by rotating R about the y-axis. Set up an integral for the volume of this solid. Evaluate the integral.

2. Another solid of revolution is made from this region, this time by rotating R about the x-axis. Set up an integral for the volume of this solid. **Do not evaluate this integral.**