Last Initial ______________________________

Section ______________________________

FULL Name ______________________________

Social Security Number ______________________________

FINAL EXAMINATION

21C, 10:30-12:30, Monday December 10, 2001

Declaration of honesty: I, the undersigned, do hereby swear to uphold the highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, books, calculators, mobile phones, immobile phones, muses or any other electronic device.

Signature ____________________ **Date** _______________

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Question 1 *(50 points)*
Let $z = 1 + i$ and $w = 2 - i$. Compute

a) $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$.

b) $z + w$, $z - w$, zw and $\frac{w}{z}$.

c) $\overline{z}$, $\overline{w}$, $\overline{z}\,\overline{w}$ and $\overline{zw}$.

Question 1 *(continued)*
d) Write z in polar form $z = re^{i\theta}$ for real r and θ. Use your result to compute z^8.

e) Now let $z = x + iy$ and $w = u + iv$ with x, y, u and v all real. Show that

$$\overline{z}\,\overline{w} = \overline{zw}\,.$$

Question 2 *(30 points)*
Compute the moment of inertia I of a solid, homogeneous sphere with radius R and total mass M about an axis through its center. *(Hint: infinitesimally $dI = \sigma r^2 dV$ where r is the distance from the axis of rotation and σ the mass density.)*

Question 3 *(20 points)*

a) Show that $\psi = f(u - v)$ satisfies

$$A\,\frac{\partial \psi}{\partial u} + B\,\frac{\partial \psi}{\partial v} = 0$$

whenever $A = B$.

b) For which values of A and B does the following equation hold

$$A\,\frac{\partial^2 \psi}{\partial u^2} + B\,\frac{\partial^2 \psi}{\partial^2 v} = 0 \text{ ?}$$

Question 4 *(20 points)*

a) Suppose $a_0 > a_1 > a_2 > a_3 .... > 0$. Does $\sum_{n=0}^{\infty} (-1)^n a_n$ converge absolutely?

YES MAYBE NO (Circle one)

b) Suppose $\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Does $\sum_{n=0}^{\infty} a_n$ diverge?

YES MAYBE NO (Circle one)

c) The series $\sum_{n=0}^{\infty}(a_n + b_n)$ converges. Does $\sum_{n=0}^{\infty} a_n$ converge?

YES MAYBE NO (Circle one)

d) Suppose we choose r such that $\sum_{n=0}^{\infty} r^n$ converges and ϵ is any positive real number. Can we find an N such that $|\frac{1-r^n}{1-r} - \frac{1}{1-r}| < \epsilon$ for all $n > N$?

YES MAYBE NO (Circle one)

e) The power series $\sum_{n=0}^{\infty} a_n x^n$ converges when $x = 1$. Does it converge absolutely when $x = .99$?

YES MAYBE NO (Circle one)

Question 5 *(30 points)*
Compute the integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}$$

by first writing $I^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(-x^2 - y^2)$ and calculating this double integral using polar coordinates.

Question 6 *(30 points)*

a) Clara the climber measures the northward slope of Mount Finality to be 3 and eastward slope to be 4. Use the total differential to estimate how much higher/lower she will be by walking 50 feet southeast.

b) Using coordinates where x measures the eastward and y the northward position and z measures height, write an equation for the tangent plane to Mount Finality at Clara's original position which you may assume to be the origin.

Question 7 *(30 points)*
The volume of a cone is $\frac{1}{3}Bh$ where B is the area of its base and h is the height. Test this result by using an iterated integral to compute the volume of the following cone: Draw a triangle in the (x, y)-plane with corners $(1, 0)$, $(1, 2)$, $(0, 2)$ and then connect every point on its edges to the point $(x, y, z) = (1, 2, 3)$.

Question 8 *(40 points)*
Consider the power series

$$f(x) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right).\left(-\frac{1}{2}\right).\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}\,(4x)^n\,.$$

a) Find the radius of convergence R. Discuss convergence at the upper endpoint $x = R$.

b) Compute the first four terms of the Taylor series for the function

$$f(x) = \frac{1 - \sqrt{1 + 4x}}{2}$$

and compare them to the power series above.

Question 8 *(continued)*
c) The series expression for $f(x)$ is an alternating series when $x > 0$. Compute an accurate estimate for $f(1/8)$ by studying the second and third partial sums. Express your answer in scientific notation $f(1/8) = (\text{answer}) \pm (\text{error})$. *(Answers using simplified fractions are OK.)*

Question 9 *(50 points)*
Let

$$f(x, y) = 1 - \sqrt{x^2 + y^2}\,.$$

a) Sketch level curves for the function f.

b) Sketch the surface $z = f(x, y)$ and its traces in the $(x, y, 0)$, $(0, y, z)$ and $(x, 0, z)$-planes.

Question 9 *(continued)*

c) For which domain of values (x, y) is the function f defined?

d) Compute the partial derivatives of f needed to determine the Hessian and critical points of f.

e) Find any extrema and critical points of f and determine their type using the Hessian where necessary to justify your answer.

Question 10 *(20 points)* Decide, using any test you think is appropriate whether the following series converge or diverge. Indicate also whether convergence is absolute or conditional.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

b) $\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$

c) $\sum_{n=1}^{\infty} \left(\frac{-3}{n}\right)^n$

d) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Question 11 *(20 points)*
Consider the sequence $a_n = \frac{1}{n^3}$. Prove that $\lim_{n\to\infty} a_n = 0$ by proposing a value N such that $0 < |a_n| < \epsilon$ for all $n > N$.

Question 12 *(30 points)*
Use the derivative form of Taylor's theorem with remainder to estimate how many terms in the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

would be needed to compute e to three decimal places. *(Note that you are not asked to compute e itself and may assume that $e < 3$.)*

Question 13 *(5 points)*
In twenty words or less: Which part of the course did you find most interesting and why?

Question 14 *(15 points)*
Let

$$f(x,y) = (1+xy) + \frac{x^2 - y^2}{x^2 + y^2}\,.$$

Explain whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

Question 15 *(30 points)*
a) Solve

$$3z^2 + 2z + 7 = 0$$

for z over $\mathbb{C}$.

b) Express $\cos(\theta)$ and $\sin(\theta)$ in terms of $e^{i\theta}$. Use these expressions to prove the trigonometrical identities

$$\cos^2\theta + \sin^2\theta = 1\,, \qquad \cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)\,.$$

Bonus Question *(3 points)*
In which years were Galileo, Leibniz and Newton born?