Vector Analysis
Math 21D, Fall 2002

Final Exam

Name: ________________________

ID: ___________________

Section: _______

- Answer all problems.
- Read each problem carefully, and show all work clearly to receive full credit.
- No calculators
- Denote all vectors with an arrow (like $\vec{A}$)
- Total points: 100.

1. *(20 points)* $f(x, y) = x^2 + 2y^2$.

(a) *(3 points)* Find the critical point(s) of f.

(b) *(12 points)* Find the maximum and minimum values of f on the circle $x^2 + y^2 = 1$.

(c) *(5 points)* Determine the absolute maximum amd minimum values of f on the disk $x^2 + y^2 \leq 1$.

2. *(20 points)* The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in a simple closed curve C (an ellipse).

(a) *(3 points)* Verify that the point $(1, 2, 1)$ lies on the curve C.

(b) *(5 points)* Find (i) a vector normal to the plane $y + z = 3$, and (ii) a vector normal to the cylinder $x^2 + y^2 = 5$ at $(1, 2, 1)$.

(c) *(6 points)* Find a vector which is tangent to the curve C at $(1, 2, 1)$.

(d) *(6 points)* Find an equation to the tangent line to the curve C at $(1, 2, 1)$.

3. *(15 points)* Let $\mathbf{F}(x,y) = \dfrac{x}{x^2+y^2}\,\mathbf{i} + \dfrac{y}{x^2+y^2}\,\mathbf{j}$ be a vector field defined on the region D between the circle (C_1), $x^2+y^2=1$, and the boundary (C_2) of the square with vertices $(2,2)$, $(-2,2)$, $(-2,-2)$ and $(2,-2)$. Both C_1 and C_2 are oriented counterclockwise.

(a) *(4 points)* Compute $(\nabla\cdot\mathbf{F})$.

(b) *(6 points)* Evaluate $\oint_{C_1} \mathbf{F}\cdot\mathbf{n}\, ds$ where $\mathbf{n}$ is the unit external normal to C_1.

(c) *(5 points)* Using your results in (a) and (b) as well as Green's theorem, evaluate $\oint_{C_2} \mathbf{F}\cdot\mathbf{n}\, ds$ where $\mathbf{n}$ is the unit external normal to C_2.

4. *(25 points)* $\mathbf{F}(x,y) = y^2\,\mathbf{i} + 2xy\,\mathbf{j}$.

(a) *(6 points)* Evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ where C_1 is the line segment from $(0,0)$ to $(1,2)$.

(b) *(8 points)* Evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where the path C_2, from $(0,0)$ to $(1,2)$, consists of two pieces: the line segment α, from $(0,0)$ to $(1,0)$ and the line segment β, from $(1,0)$ to $(1,2)$.

(c) *(5 points)* Using Green's theorem, show that the vector field $\mathbf{F}(x,y)$ is conservative: $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for **any** simple closed curve in the XY-plane.

(d) *(6 points)* Construct a scalar field $f(x,y)$ such that $\mathbf{F}(x,y) = \nabla f(x,y)$.

5. *(20 points)* Let $\mathcal{S}$ be the surface defined by $z = f(x,y) = 2-y$ where (x,y) is on the disk (D), $x^2+y^2 \le 1$, in the z = 0 plane. Thus, $\mathcal{S}$ is a portion of the plane $y+z=2$, bounded by a simple closed curve C. Let $\mathbf{F}(x,y,z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ be a vector field on $\mathcal{S}$.

(a) *(3 points)* Compute $(\nabla\times\mathbf{F})$.

(b) *(12 points)* Orient $\mathcal{S}$ such that the induced orientaion on its boudary C is counterclockwise when viewed from above. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, using Stokes' theorem.

(c) *(5 points)* Using the formula, (Area of $\mathcal{S}$) $= \int_{\mathcal{S}} 1\, dS$, find the area of $\mathcal{S}$.