MATH 21 D (Section A) NAME: ______________________

Quarter: Spring SECTION: ______________________

Date: June 08, 2002

1	2	3	4	5	total	6

Final

Problem 1. (30 pts; *estimated time:* **20 mn**)
Find the dimensions of the box of **largest volume** whose surface area is to be 6 square inches (see figure 1).

Problem 2. (40 pts; *estimated time:* **25 mn**)

Denote by $\mathcal{S}$ the surface of equation: $z = -\frac{1}{2}x^2 + y^2 + 2x + 1$.

(a) Verify that the point $(0, 1, 2)$ belongs to the surface $\mathcal{S}$.

(b) Verify that the vector $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ is tangent to the surface $\mathcal{S}$ at the point $(0, 1, 2)$.

(c) Find the parametric equations of the **vertical plane** passing through the point $(0, 1, 2)$ and containing the vector $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

(d) Find the parametric equations of a **curve drawn on the surface** $\mathcal{S}$ whose tangent at the point $(0, 1, 2)$ is the vector $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Problem 3. (40 pts; *estimated time:* **25 mn**)

(a) Show that the curve parameterized by

$$\mathbf{G}(t) = \frac{e^t + e^{-t}}{2}\mathbf{i} + \frac{e^t - e^{-t}}{2}\mathbf{j} = \cosh t\mathbf{i} + \sinh t\mathbf{j},$$

for $0 \leq t \leq a$, with $a > 0$, lies on the parabola $x^2 - y^2 = 1$, and joins the point $A = (1, 0)$ to the point $B = (\cosh a, \sinh a)$.

(b) Let $\mathcal{A}$ be the region bounded by the line segment OA, the line segment OB, and the curve in (a) joining A to B (figure 2.) Show that the area of $\mathcal{A}$ is $a/2$.

Problem 4. (60 pts; *estimated time:* **30 mn**)
Let

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i}}{x^2 + y^2} + \frac{x\mathbf{j}}{x^2 + y^2}.$$

(a) Show that the **divergence** of $\mathbf{F}$ is zero.

(b) Show that the **curl** of $\mathbf{F}$ is zero.

(c) Let $f(x, y) = \tan^{-1}(y/x)$. Does $\mathbf{F}$ equal ∇f where both are defined? **Justify your answer**.

(d) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 1$ in the xy plane, taken counterclockwise.

(e) Is $\mathbf{F}$ **conservative**? Justify your answer.

(f) Does (e) contradict (c)? **Explain**.

Problem 5. (30 pts; *estimated time:* **20 mn**)
Let $\mathbf{F}$ be defined everywhere in the space except on the z axis. Assume that $\mathbf{F}$ is irrotational, that is, $\nabla \times \mathbf{F} = 0$. Denote by C_1 the circle of center $(0, 0, 2)$ of radius 1 contained in the plane $z = 2$, C_2 the circle of center $(0, 0, 3)$ of radius 1 contained in the plane $z = 3$, C_3 the circle of center $(0, 1/2, 1/2)$ of radius 1 contained in the plane $y + z = 1$, and C_4 the circle of center $(0, 3, 0)$ of radius 1 contained in the plane $y = 3$. (see figure 3 for orientations). Suppose that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 3$. Find the values of the following integrals

(a) $\displaystyle\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ (b) $\displaystyle\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$ (c) $\displaystyle\oint_{C_4} \mathbf{F} \cdot d\mathbf{r}$.

Hint: Take the surface of cylinder bounded by the circles C_1 and C_2 and apply Stokes's theorem (careful for orientations). Use cancellation principle for (b).

Problem 6. (Extra credit: 20 pts)

(a) Let $\mathcal{S}$ be the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Using the formula

$$\text{Area of } \mathcal{S} = \int_{\mathcal{S}} 1 dS,$$

find the area of $\mathcal{S}$.

(b) If $\mathbf{F}$ is a vector field in the plane, divergence-free, what is the flux of $\mathbf{F}$ across the circle of radius 1 centered at the origin and oriented clockwise. Justify.

NB: Problem 6 is an extra credit: you are not supposed to do it.