

**MATH 22B**  
**ORDINARY DIFFERENTIAL EQUATIONS**  
**FINAL EXAMINATION**

Full Name: \_\_\_\_\_

Instructions:

1. Write your name at the top of each page.
2. You may use only the following resources: a pencil, an eraser, your brain.
3. Solve each problem in the space provided, showing your work *neatly* and *legibly*. Partial credit cannot be given for unintelligible responses. If you need to continue your solution on the back of a page, indicate that you are doing so.
4. Express all numbers in exact arithmetic. Do not use decimal approximations.
5. Read and sign the following honor code pledge.

Upon my honor, I have neither given nor received any aid on this examination.

X \_\_\_\_\_

**Problem 1. (20 points).** Solve the following initial value problem.

$$\frac{dx}{dt} + tx = e^{-\frac{t^2}{2}}, \quad x(0) = 1$$

**Problem 2. (20 points).** Recast the following fourth-order equation as a system of first-order ODE.

$$\frac{d^4y}{dt^4} - 3t \frac{d^2y}{dt^2} + \sin(2t)y = 7e^{-t}$$

**Problem 3.** Each of the following statements is either true or false. In the blank provided, indicate which of these is the case. Do not justify your answer. [NB: A correct answer scores 4 points; an incorrect answer, -2 points.]

- (a) \_\_\_\_\_ The initial value problem

$$\frac{dx}{dt} = x^2 - |t|, \quad x(0) = 0$$

Has a unique solution over some interval  $\alpha < t < \beta$ , where  $\alpha < 0 < \beta$ .

- (b) \_\_\_\_\_ Suppose  $x_1(t)$ ,  $x_2(t)$  are two solutions of the ODE

$$\frac{d^2x}{dt^2} + \frac{1}{1+t^2} \frac{dx}{dt} + (1-t^2)x = 0.$$

Suppose also that  $x_1(0)x_2'(0) - x_2(0)x_1'(0) \neq 0$ . Then  $x_1(t)x_2'(t) - x_2(t)x_1'(t) \neq 0$  for all  $t$ .

- (c) \_\_\_\_\_ If  $x_1$  and  $x_2$  are any two solutions of the equation

$$t^2 \frac{d^2x}{dt^2} + (1-x) \frac{dx}{dt} + 4x = 0,$$

then  $x_1 + x_2$  is also a solution.

- (d) \_\_\_\_\_ If  $F(s)$  is the Laplace transform of  $f(t)$ , then

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt.$$

- (e) \_\_\_\_\_ The initial value problem

$$\begin{aligned} \frac{dx}{dt} &= -3x + 4ty - 8t^2, & x(0) &= 2 \\ \frac{dy}{dt} &= 9t^2x - 3y, & y(0) &= 0 \end{aligned}$$

has a unique solution defined for all  $t \geq 0$ .

**Problem 4. (20 points).** Consider the autonomous first-order ODE

$$\frac{dx}{dt} = ax - x^3. \quad (*)$$

- (a) Suppose  $a < 0$ . Find all equilibria and tell whether each is asymptotically stable, unstable, or semistable. Sketch the one-dimensional vector field corresponding to the ODE.
- (b) Suppose  $a > 0$ . Find all equilibria and tell whether each is asymptotically stable, unstable, or semistable. Sketch the one-dimensional vector field corresponding to the ODE.
- (c) Let  $x = \phi(t, a, x_0)$  be the solution of (\*) with initial condition  $x(0) = x_0$ . In other words,

$$\frac{\partial \phi}{\partial t}(t, a, x_0) = a \phi(t, a, x_0) - \phi(t, a, x_0)^3 \quad \text{and} \quad \phi(0, a, x_0) = x_0,$$

for all values of  $t$ ,  $a$ , and  $x_0$ . Describe the dependence of

$$\lim_{t \rightarrow \infty} \phi(t, a, x_0)$$

on  $a$  and  $x_0$ .

**Problem 5. (20 points).** Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}.$$

You may use the fact that

$$e^{At} = \begin{pmatrix} \cos t + 2 \sin t & -5 \sin t \\ \sin t & \cos t - 2 \sin t \end{pmatrix}.$$

*Hint:* Assume a solution of the form  $\mathbf{x}(t) = e^{At} \mathbf{u}(t)$  and solve for  $\mathbf{u}(t)$ .

**Problem 6. (40 points).** Consider the initial value problem (IVP)

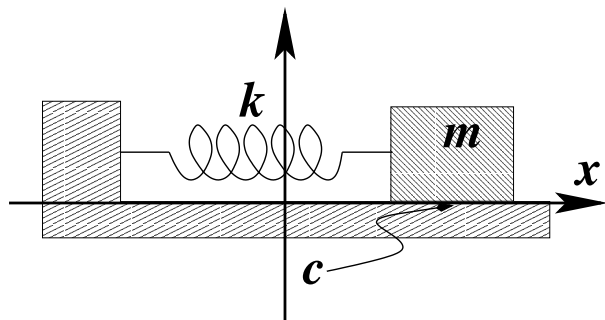
$$y'' - 4y' + ky = f(t), \quad y(0) = 0, \quad y'(0) = 6.$$

- (a) Let  $y(t)$  be the solution of the IVP and  $Y(s)$  its Laplace transform. Solve for  $Y(s)$ .
- (b) Suppose  $k = -5$  and  $f(t) = 0$ . Find  $y(t)$ .
- (c) Suppose  $k = 4$  and  $f(t) = 0$ . Find  $y(t)$ .
- (d) Suppose  $k = 13$  and  $f(t) = 0$ . Find  $y(t)$ .
- (e) Suppose  $k = 4$  and  $f(t) = 2e^{2t}$ . Find  $y(t)$ .

**Problem 7. (20 points).** An iron box of mass  $m$ , connected to a spring (spring constant  $k$ ), slides on a table with coefficient of friction  $c$ . Thus, the force of friction is given by

$$F_{fric} = -c m g v,$$

where  $v$  is the velocity of the box and  $g$  is the local gravitational field. The box is drawn so as to extend the spring to a length  $x_0$  beyond its natural length. The box is then released.



- (a) Use Newton's second law, applied to the forces in the horizontal direction, to write down a differential equation for  $x(t)$ . Write down the appropriate initial conditions.
- (b) How large must  $m$  be so that the system is overdamped? That is, how massive must the box be in order that it returns to rest without changing its direction of motion?
- (c) Assuming that the system is underdamped, derive a solution to the differential equation of part (a).

**Problem 8. (40 points).** Consider the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (*)$$

where  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  and  $A = \begin{pmatrix} -1 & -1 & 2 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{pmatrix}$ .

- (a) Show that there are only two eigenvalues of  $A$  and that they are  $\lambda_1 = 0$ ,  $\lambda_2 = -3$ .
- (b) Compute eigenvectors  $\mathbf{v}^1$ ,  $\mathbf{v}^2$  corresponding to  $\lambda_1$ ,  $\lambda_2$ , respectively.
- (c) Compute a generalized eigenvector,  $\mathbf{v}^3$  corresponding to  $\lambda_2$  by solving

$$(A - \lambda_2 I) \mathbf{v}^3 = \mathbf{v}^2.$$

- (d) Find the solution  $\mathbf{x} = \mathbf{x}(t)$  of the IVP (\*).
- (e) Evaluate  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$ .

## APPENDIX A. TABLE OF LAPLACE TRANSFORMS

Function	Transform	Conditions
$f(t)$	$\mathfrak{L}[f(t)](s)$	
$f(t)$	$F(s)$	
1	$\frac{1}{s}$	
$t^n$	$\frac{n!}{s^{n+1}}$	$n$ a positive integer
$e^{\alpha t}$	$\frac{1}{s - \alpha}$	$s > \alpha$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$s > 0$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$s > 0$
$\sinh \alpha t$	$\frac{\alpha}{s^2 - \alpha^2}$	$s >  \alpha $
$\cosh \alpha t$	$\frac{s}{s^2 - \alpha^2}$	$s >  \alpha $
$a f(t) + b g(t)$	$a F(s) + b G(s)$	
$f(t - c)$	$e^{-cs} F(s)$	$f(t) = 0$ for $t < 0$
$f(\alpha t)$	$\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$	$\alpha > 0$
$e^{\alpha t} f(t)$	$F(s - \alpha)$	
$t f(t)$	$-F'(s)$	
$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(s)$	$n$ a positive integer
$f'(t)$	$s F(s) - f(0)$	
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	
$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$	
$\int_0^t f(t - \tau) g(\tau) d\tau$	$F(s) G(s)$	