Last Initial ______________________________

FULL Name ______________________________

Social Security Number ______________________________

FINAL EXAMINATION

22B, 3:10-4:00 pm, Friday March 22, 2002

Declaration of honesty: I, the undersigned, do hereby swear to uphold the highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, books, calculators, mobile phones, bread machines, mp3 players, handheld toaster ovens, razor scooters, London buses, personal cooling systems or any other electronic device.

Signature ____________________ **Date** _______________

1	2	3	4	5	6	TOTAL

Advice

This examination is divided into two parts: Part A asks you to employ the methods studied during the course to solve various initial value problems. Do it first and carefully – emphasis will be placed on correct answers. Part B involves harder questions where you can display your understanding of the material. Partial credit will only be assigned for steps that genuinely contribute to the solution and will be limited. Therefore, you are advised to use your time doing a few questions well, rather than aiming for partial credit for all questions.

A Skills Questions

Question 1 *(100 points)* Solve the following initial value problems using your favorite method. In each case, indicate whether your solution is the unique one. If it is not, explain why and find all solutions. *The majority of the credit will be awarded for the correct answer with some work. If practical, check that each of your solutions satisfies the original differential equation and initial conditions.*

1. $$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}\,, \qquad y(1) = 1$$

2. $$\frac{dy}{dx} = \frac{x}{y+x} + \frac{y}{x}\,, \qquad y(1) = -1$$

3. $$\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + y = \sin(x)\,, \qquad y(0) = 0\,, \quad y'(0) = 1$$

4. $$\begin{cases} \frac{dk}{dt} = k + 2m\,, & k(0) = 1 \\ \frac{dm}{dt} = -k + 4m\,, & m(0) = 0 \end{cases}$$

Question 1 working

Question 1 still working...

Question 1 still still working...

B Concept Questions

Question 2 *(100 points)* Raindrops keep falling on my head. Worse still, its very misty so they get bigger as they fall (the scientific term is accretion). In this problem we will show (under simplifying assumptions) that the acceleration of an accreting raindrop is $g/7 \text{ m.s}^{-2}$.

1. Introduce some variables: call $M(t)$ kg the raindrop's mass at time t s. Assume that it remains spherical at all times so call its radius $R(t)$ m and use the variable $v(t) \text{ m.s}^{-1}$ for its velocity. Draw a picture of the raindrop showing all the variables and the forces acting on it.

2. Now we need an equation describing its motion. The momentum of the raindrop is the product of its mass and velocity, $M(t)v(t) \text{ kg.m.s}^{-1}$. Isaac Newton says that the rate of change of momentum is equal to the sum of the forces. Assume that air resistance is negligible so the only force is that of gravity $M(t)g \text{ kg.m.s}^{-2}$. Translate this statement into a differential equation involving $M(t)$, $v(t)$ and g.

3. Water is incompressible, so we can assume that the mass density of the drop is constant. Write down the relation between the mass and radius of the spherical raindrop. Call the constant of proportionality k_1 $\mathrm{kg.m}^{-3}$.

4. Lets assume that the rate of change of the mass of the raindrop is proportional to its cross sectional area multiplied by its velocity. Translate this statement into a differential equation. Call this constant of proportionality k_2 $\mathrm{kg.m}^{-3}$.

5. Combine the equations you found in parts 3 and 4 above to produce a simple equation relating the rate of change of the radius to the velocity.

6. Now we want to write the equation of motion found in part 2 in terms of the variables $v(t)$ and $R(t)$ only. Do this using: (i) The mass-radius relation found in part 3 to eliminate $M(t)$. (ii) Your result for the rate of change of the mass in part 4. (iii) The chain rule along with the result for the rate of change of the radius found in part 5 to eliminate any remaining time derivatives.

7. If all went well you now have a differential equation involving the variables v and R (and a bunch of constants) in which v is now regarded as an unknown function of the independent variable R. Show that this equation is linear in the variable v^2.

8. Solve the differential equation for v^2 found in part 7. As initial condition, assume that the velocity is zero when the radius of the raindrop vanishes (raindrop formation).

9. The last step. Return to the equation of motion in part 2 because it tells you the rate of change of the velocity (or in other words ACCELERATION). You already know how to eliminate the mass and its rate of change (using parts 3 and 4), but now you can also substitute your solution for the velocity in terms of the radius from part 8. Then you can solve for the acceleration and, bloopers notwithstanding, are now in seventh heaven!

Question 3 *(40 points)* Psychologists studying mathematics students noticed the following strange phenomena:

1. If the students knew no calculus and then were locked in a padded cell with a copy of Barcellos and Stein, their knowledge of calculus remained zero for all later times.

2. If the students knew a little bit of calculus, (*i.e.* less than that in Barcellos and Stein), then after sufficiently long in the padded cell, their knowledge of calculus was very close to that in the textbook.

3. Students whose calculus knowledge coincided with the contents of Barcellos and Stein, maintained that level for all later times.

4. Students who knew more calculus than the contents of Barcellos and Stein eventually reverted back to a knowledge level close to that of the textbook.

Develop a mathematical model describing the psychologist's results based on a first order autonomous ordinary differential equation. (*Your answer should include two graphs, the first representing the information in points 1-4 above, the second sketching any functions of the unknown variable that appear in your proposed differential equation*.)

Question 3 working

Question 4 *(80 points)*

a) Sketch the direction field in the (k, m)-plane for the system of equations in part 4 of question 1. Draw several trajectories for various initial conditions including the one you obtained explicitly in question 1. **b)** Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

c) Solve the IVP

$$\begin{cases} \dfrac{dk}{dt} = k + 2m + \exp(t)\,, & k(0) = 1 \\ \dfrac{dm}{dt} = -k + 4m + \exp(t)\,, & m(0) = 0 \end{cases}$$

(*Hint, try to make the problem diagonal in terms of a new unknown vector variable*.)

Question 5 *(60 points)*

a) If $y_1(t)$ and $y_2(t)$ are solutions to

$$y'' + p(t)y' + q(t)y = 0,$$

what is a criterion for them to be linearly independent?

b) Suppose the Wronskian $W[y_1, y_2](t) \neq 0$ and initial conditions are $y(0) = y_0$, $y'(0) = y'_0$. Prove that you can always find constants c_1 and c_2 such that the solution $c_1 y_1(t) + c_2 y_2(t)$ satisfies the initial value problem.

c) Rewrite the second order differential equation in part a) as a pair of first order equations. This system has two solutions. When are they independent? Show that your answer is equivalent to the one for part a).

Question 6 *(80 points)*

a) Compute

$$\frac{d}{dt}\sin(t)\,, \qquad \frac{d}{dt}\sinh(t) = \frac{d}{idt}\sin(it)\,,$$

$$\frac{d^2}{dt^2}\sin(t) \qquad \text{and} \qquad \frac{d^2}{dt^2}\sinh(t)\,.$$

b) The Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}[f] \equiv \int_0^\infty dt\ \exp(-st)\ f(t)\,.$$

Compute

$$\mathcal{L}[\sin(t)] \qquad \text{and} \qquad \mathcal{L}[\sinh(t)]\,.$$

Now we study the initial value problem

$$\frac{d^4y}{dt^4} = y\,, \qquad\qquad y(0) = 0,\; y'(0) = 1\,, y''(0) = 0\,, y'''(0) = 0\,.$$

c) What is $y''''(0)$?

d) Solve the initial value problem using the Laplace transform method.

Question 6 working