

FINAL TEST - 25

Name and ID number:

**Do not turn this page until instructed to do so**

**Instructions:** Read carefully every problem. Show your work on every problem. Correct answers with no support work will not receive full credit. Be organized and use notation appropriately.

NO notes, books, cellphones, ipods, etc. may be used on this exam. You should have only a pencil, eraser and your student ID on your desk. Please write legibly. Recall to silent your cell phone.

**Have your ID on your desk. NO CHEATING.**

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Problem	Points possible	Points received
1	20	
2	20	
3	15	
4	10	
5	20	
6	20	
7	20	
8	10	
9 extra credit	20	
total	135	

**Problem 1** (20 points)

Investigate the behavior of the following series. JUSTIFY your answer. (4pt each part).

(a)  $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

(b)  $\sum_{n=1}^{\infty} 2^n/n!$

(c)  $\sum_{n=1}^{\infty} (-1)^n/3n^2$

(d)  $\sum_{n=1}^{\infty} (-3/13)^n$

(e) Use the fact that  $\sum_{n=1}^{\infty} (n-1)/(2^{n+1}) = 1/2$  to find  $\sum_{n=1}^{\infty} n/2^n$

**Problem 2** (20 points)

Let  $(s_n)$  be the sequence defined as:  $s_1 = \sqrt{2}$  and  $s_{n+1} = \sqrt{2 + s_n}$ .

- (a) Prove that  $(s_n)$  is a nondecreasing sequence. (By induction)
- (b) Show that  $s_n \leq 2$  for all  $n$ . (By induction)
- (c) Show that the limit of  $s_n$  exists.
- (d) Find  $\lim_{n \rightarrow \infty} s_n$

**Problem 3** (15 points)

Consider the sequence  $(s_n)$  given by  $s_n = (1 + (-1)^n)e^{1/n}$ . Find:

- (a)  $\limsup s_n$
- (b)  $\liminf s_n$ .
- (c) Find a subsequence converging to  $\limsup s_n$ .

(Hint: It'll be helpful if you write some terms of the sequence. Useful fact:  $\lim_{x \rightarrow 0} e^x = 1$ ).

**Problem 4** (10 points)

Why is the following argument wrong? EXPLAIN using your words/definitions/theorems.

$$\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = \lim_{n \rightarrow \infty} \sin(n) \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} \sin(n) \cdot 0 = 0$$

**Problem 5** (20 points)

(a) Give the definition of a metric.

(b) Let  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  be points in  $\mathbb{R}^2$ . Define the function:

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$$

Show that  $d$  is a metric for  $\mathbb{R}^2$ . ( Hint: Use the properties of the absolute value)



**Problem 7**(20pt)

- (a) Give the formal definition for a series  $\sum_{n=1}^{\infty} a_n$  to satisfy the Cauchy criterion.
- (b) Negate your answer to part (a) to give a formal definition for a series  $\sum_{n=1}^{\infty} a_n$  not to satisfy the Cauchy criterion.
- (c) If  $\sum |a_n|$  converges and  $(b_n)$  is a bounded sequence, then  $\sum a_n b_n$  converges. (Hint: Prove that  $\sum a_n b_n$  satisfies the Cauchy criterion and conclude that it converges.)

**Problem 8**(10pt)

State the following theorems:

(a) Archimidean Property

(b) Bolzano-Weierstrass Theorem (in  $\mathbb{R}$ ).

**Extra credit: Problem 9** (20 points)

True or False, no justification is needed.

- 1(    ) A set  $E$  is closed if and only if  $E = \bar{E}$ .
- 2(    ) If  $\lim a_n = 0$  then  $\sum a_n$  converges.
- 3(    ) A convergent sequence is bounded.
- 4(    ) Let  $(\mathbf{x}^{(n)})$  be a sequence in  $\mathbb{R}^k$  such that  $d(\mathbf{x}^{(n)}, 0) \leq 3$  for all  $n$ , then  $(\mathbf{x}^{(n)})$  has a subsequence that converges in  $\mathbb{R}^k$ .
- 5(    )  $\mathbb{Q}$  is a complete space.
- 6(    )  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .
- 7(    )  $\mathbb{R}^k$  with the euclidean metric is open and closed.
- 8(    )  $\overset{\circ}{A} \subset A \subset \bar{A}$ ,  $\overset{\circ}{A}$  denotes the interior of the set  $A$  and  $\bar{A}$  its closure.
- 9(    ) The set  $\{x^2 | x \in \mathbb{R}\}$  is bounded above.
- 10(    ) If  $S \subset \mathbb{R}$  is nonempty and bounded, then  $\inf S \leq \sup S$ .