

PRELIMINARY EXAM
Math 119a
Fall 2000

Section One (of Two)
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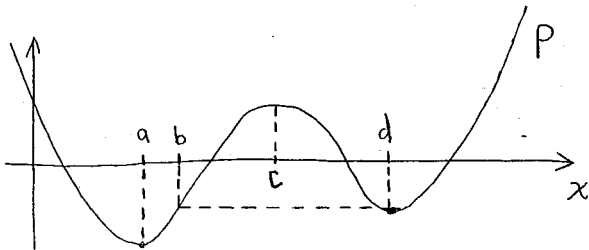
Problem 1. Find $\exp(A)$ in the following cases:

(i) $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Problem 2. Consider solutions to the ODE $\ddot{x} = -P'(x)$, where $P(x)$, the potential energy, is given in graphical form below. Use the graph of P to graph the phase portrait for this ODE, (that is, graph the trajectories in the phase space, the $x\dot{x}$ -plane), making sure to accurately plot the points $a - d$.



Problem 3.(Chaos and the Lorentz Equations) The Lorentz Equations are given by

$$\dot{x} = -\sigma x + \sigma y, \quad (1)$$

$$\dot{y} = -xz + rx - y, \quad (2)$$

$$\dot{z} = xy - bz, \quad (3)$$

where σ , r and b are positive constants. (Lorentz used $\sigma = 10$, $b = 8/3$, $r = 28$.) Lorentz introduced this equation, (a simplified model for convection in the atmosphere), in the mid-sixties, and this is recognized as the first system of ODE's for which chaotic behavior could be rigorously demonstrated, (thereby suggesting that it is very difficult to predict the weather!)

(i) Prove that when $0 < r < 1$, there exists only one rest point $(0, 0, 0)$, and all solution trajectories tend to this stable rest point as $t \rightarrow \infty$. Prove this by stating the Liapunov stability theorem, and then showing that $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$ is a Liapunov function when $0 < r < 1$. (That is, $\frac{d}{dt}V(\mathbf{x}(t)) < 0$ for $(x, y, z) \neq (0, 0, 0)$, and thus all solution trajectories head toward the rest point $(0, 0, 0)$ as $t \rightarrow \infty$.) It follows that system (2) goes from *predictable* to *chaotic* as parameter values change from $r < 1$ to $r > 1$.

(ii) One of the main forces that drives the chaotic behavior of the Lorentz equations is that individual solution trajectories diverge from one another at an exponential rate, but the volume of any region is exponentially squashed, under the dynamics. But this wouldn't be such an interesting set of constraints if solution trajectories were unbounded, since then they could just go off to infinity. Thus, a main step in the analysis of (2) is to prove that solutions remain bounded for all time. Prove this by showing that solutions of (2) starting inside the ellipsoid $E = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$, must remain inside of E so long as C is large enough so that E contains the ellipsoid $\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} \leq 1$. (Hint: Show that E defines a Liapunov function that decreases on solutions.)

GGAM PRELIM QUESTIONS
Fall, 2000Section Two (of Two)
Page 1**Problem 1.** Compute the Green's function for the BVP

$$\begin{aligned}u'' + u &= f & 0 < x < 1, \\u(0) &= a, & u'(1) = b,\end{aligned}$$

where $f : [0, 1] \rightarrow \mathbb{R}$ is a given continuous function and a, b are given real constants. Write out the Green's function representation of the solution u .

Problem 2. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given bounded, continuous functions and $\lambda \in \mathbb{R}$. Prove that if $|\lambda| < 1/2$, then there is a unique bounded, continuous solution $u : \mathbb{R} \rightarrow \mathbb{R}$ of the nonlinear integral equation

$$u(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-y|} \sin[u(y) - g(y)] dy = f(x).$$

Problem 3. Let (f_n) , (g_n) , (h_n) be the sequences of functions in $L^2(\mathbb{R})$ defined by:

$$\begin{aligned}f_n(x) &= \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases} \\g_n(x) &= \begin{cases} 1/n & \text{if } 0 < x < n, \\ 0 & \text{otherwise;} \end{cases} \\h_n(x) &= \begin{cases} 1 & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In each case, determine (with proof) whether or not the sequence converges: (a) strongly in $L^2(\mathbb{R})$; (b) weakly in $L^2(\mathbb{R})$; (c) in the sense of distributions.

Problem 4. Let $A : \mathcal{D}(A) \subset L^2([0, 1]) \rightarrow L^2([0, 1])$ be the differential operator

$$\begin{aligned}Au &= (a - c)(-u'' + u) + a''u, \\ \mathcal{D}(A) &= \{u : [0, 1] \rightarrow \mathbb{C} \mid u \in H^2([0, 1]), u(0) = u(1) = 0\},\end{aligned}$$

where $a : [0, 1] \rightarrow \mathbb{R}$ is a given real-valued, twice continuously-differentiable function, and $c \in \mathbb{C} \setminus \mathbb{R}$ is a complex constant with nonzero imaginary part.

- (a) Compute the adjoint A^* of A .
(b) If $Au = 0$, show that

$$\int_0^1 \left(|u'|^2 + |u|^2 + \frac{a''}{a-c} |u|^2 \right) dx = 0.$$

Deduce that if a'' does not change sign in the interval $[0, 1]$, meaning that a has no inflection points, then the kernel of A is $\{0\}$.

Problem 5. Let $\Omega = \{(r, \theta) \mid r < 1\}$ be the unit disc in the plane, where (r, θ) are polar coordinates. The boundary of Ω is the unit circle \mathbb{T} . Let $\mathcal{H} \subset L^2(\mathbb{T})$ be the Hilbert space

$$\mathcal{H} = \left\{ f \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} f(\theta) d\theta = 0 \right\}.$$

We define a map $N : \mathcal{H} \rightarrow \mathcal{H}$ in the following way: for $f \in L^2(\mathbb{T})$, let $u(r, \theta)$ be a solution of Laplace's equation in Ω ,

$$\frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad r < 1,$$

such that $u_{\theta}(1, \theta) = f(\theta)$. Then $Nf = g$ where $g(\theta) = u_r(1, \theta)$. Thus, N maps the θ -derivative of the Dirichlet data for u to the Neumann data for u . Prove that N is a well-defined, unitary map on \mathcal{H} .

Problem 6. The Wigner distribution $W : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ of a Schwartz function $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ is defined by

$$W(x, k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi \left(x - \frac{y}{2} \right) \overline{\varphi \left(x + \frac{y}{2} \right)} e^{ik \cdot y} dy,$$

where $x, k \in \mathbb{R}^n$.

- (a) Prove that

$$\int_{\mathbb{R}^n} W(x, k) dk = |\varphi(x)|^2.$$

- (b) Prove that

$$W(x, k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi} \left(k - \frac{\ell}{2} \right) \overline{\hat{\varphi} \left(k + \frac{\ell}{2} \right)} e^{-i\ell \cdot x} d\ell$$

where

$$\hat{\varphi}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ik \cdot x} dx$$

is the Fourier transform of φ .