

Explain all your answers unless otherwise instructed. State precisely or indicate by name the theorems you use in your arguments.

1. (a) Prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon^c} \frac{1}{|x|} \Delta f(x) dx = 4\pi f(0), \quad \forall f \in S(\mathbb{R}^3)$$

where B_ε^c is the complement of the ball of radius ε centered at the origin.

- (b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(x), \quad \lim_{|x| \rightarrow \infty} u(x) = 0$$

for $f \in S(\mathbb{R}^3)$.

2. Consider the following sequences of functions parametrized by n .

$$f_n(x) = e^{i(2\sqrt{n}\pi x)}, \quad \text{in } L^2([0, 1]) \quad (1)$$

$$f_n(x) = \sqrt{n}e^{-nx^2}, \quad \text{in } L^2(\mathbb{R}) \quad (2)$$

$$f_n(x) = \sqrt{n}e^{-n|x|^2}, \quad \text{in } L^2(\mathbb{R}^2) \quad (3)$$

$$f_n(x) = ne^{-nx^2}, \quad \text{in } L^2(\mathbb{R}) \quad (4)$$

$$f_n(x) = ne^{-n|x|^2}, \quad \text{in } L^2(\mathbb{R}^2) \quad (5)$$

$$f_n(x) = \sum_{k=-n}^{k=n} e^{i2k\pi x}, \quad \text{in } L^2([0, 1]) \quad (6)$$

$$f_n(x) = e^{-(x-n)^2}, \quad \text{in } L^2(\mathbb{R}) \quad (7)$$

As n tends to infinity, which sequences converge (a) almost everywhere (b) L^2 -strongly (c) L^2 -weakly but not strongly (d) in distribution but not L^2 -weakly?

3. Consider the following operator $A_1 : D(A_1) \subset L^2([0, 1]) \rightarrow L^2([0, 1])$, acting on complex-valued functions u , defined by

$$A_1 u = \frac{1}{i} \frac{du}{dx}$$

with the domain

$$D(A_1) = \left\{ u \in H^1([0, 1]) : u(0) = u(1) \right\}.$$

- (a) Is A_1 bounded? self-adjoint? compact? Explain.
 (b) Compute the eigenvalues and eigenfunctions of A_1 .
4. (a) Does the Fourier series

$$g(x) := \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{2n\pi} e^{i2n\pi x}$$

define a function $g \in L^2([0, 1])$? Explain.

- (b) Consider the following operator $A_2 : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined by

$$A_2 f = \int_0^1 g(x-y)f(y)dy, \quad f \in L^2([0, 1]).$$

Is A_2 bounded? self-adjoint? compact? Explain.

- (c) Find the eigenvalues and eigenfunctions of A_2 . (Hint: compare with A_1 in the previous problem)
5. Which of the following statements are false? You do not need to explain your answers for this problem.

- (a) The dual (as Banach space) of $L^1([0, 1])$ is $L^\infty([0, 1])$.
 (b) The dual (as Banach space) of $L^\infty([0, 1])$ is $L^\infty([0, 1])$.
 (c) Dirac's delta function $\delta(x)$ is in $L^1(\mathbb{R})$.
 (d) The Fourier transform is an isometry on $L^p(\mathbb{R})$, $1 \leq p < \infty$.
 (e) The Fourier transform has a complete set of orthonormal eigenfunctions in $L^2(\mathbb{R})$.
 (f) For all $f \in L^1(\mathbb{R})$, $\hat{f}(k) \rightarrow 0$ as $k \rightarrow \infty$.
 (g) For all $f \in L^2(\mathbb{R})$, $\hat{f}(k) \rightarrow 0$ as $k \rightarrow \infty$.
 (h) Suppose $f_n \geq 0$ converges point-wise decreasingly to f . Then $\lim_{n \rightarrow \infty} \int f_n(x) dx = \int f(x) dx$.
 (i) Every norm-bounded sequence in $L^2([0, 1])$ has a L^2 -strongly convergent subsequence.
 (j) Every norm-bounded sequence in $H^1([0, 1])$ has a L^2 -strongly convergent subsequence.

6 Assume $\underline{x}(t) = (x_1(t), x_2(t))$ satisfies the DE

$$\begin{aligned}\frac{dx_1}{dt} &= 2x_1 + x_2 + x_1x_2 \\ \frac{dx_2}{dt} &= -3x_1 + (x_2)^2\end{aligned}$$

Find the matrix A such that the linearization of this system at $(0,0)$ is given by

$$\frac{d\underline{x}}{dt} = A\underline{x}.$$

7 Compute $\exp(At)$ where

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

for $\lambda \neq 0$.

8 (a) Prove that

$$E = \frac{1}{2}\dot{x}^2 + P(x)$$

is constant along solutions of

$$\ddot{x} = -P'(x).$$

(b) Consider the 2nd order ODE

$$\ddot{x} = -\cos x + Lx \quad (*)$$

Find a first order scalar ODE that describes the solutions of $(*)$. Argue that (in general) this first order equation requires the same number of initial conditions as $(*)$. Demonstrate that solutions of the scalar 1st order equation are indeed (in general) solutions of $(*)$.

9 Let A be an $n \times n$ diagonalizable matrix of constants, and assume that the eigenvalues of A all have negative real part. Prove that

$$\lim_{t \rightarrow \infty} \underline{x}(t) = 0$$

for any solution of

$$\dot{\underline{x}} = A\underline{x}.$$