APPLIED MATHEMATICS
PRELIMINARY EXAMINATION
Winter 1995

Put your answers to questions 1–3, 4–7, and 8–9 into separate piles. Make sure your name is on each pile.

1. Two stationary charged particles p_1 and p_2 are located at distance 1 from each other. A third particle p_3 is constrained to move along the line segment joining p_1 and p_2, hence its position is determined by $x \in (0,1)$, the distance from p_1. The equation of motion is determined by the inverse–square law

$$x'' = F(x),$$

where

$$F(x) = -\frac{A}{(1-x)^2} + \frac{B}{x^2}.$$

Assume that all particles are positively charged $(A, B > 0)$ so that the forces between them are repulsive.

(a) Write down the equivalent first order system.

(b) Find the equilibrium position and velocity of the particle p_3.

(c) Draw the global phase portrait.

(d) Does the linearization of the system around the equilibrium correctly predict the system's local behavior near the equilibrium? Explain your answer.

(e) What is the behavior of the trajectories as $t \to \infty$ if a small amount of friction is introduced, i.e.

$$x'' = F(x) - \epsilon x',$$

where $\epsilon > 0$ is small?

2. Consider the system

$$x' = x(3 - 2x - y)$$
$$y' = y(2 - x - y)$$

with initial conditions $x(0) = a$, $y(0) = b$, where $a, b > 0$ and $(a, b) \neq (1, 1)$. Show that the limit

$$\lim_{t\to\infty} \frac{1 - y(t)}{1 - x(t)}$$

always exists and determine its possible values.

3. Find the fixed points of the following system and for each of them determine whether it is asymptotically stable, neutrally stable, or unstable.

$$x' = xy - 1$$
$$y' = x - y^3$$

4. Define $L : D_L \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{aligned} Lu &= -(e^x u')' + xu, \\ D_L &= \{u \in C^\infty[0,1] : u'(0) = u'(1) = 0.\}. \end{aligned}$$

(a) Prove that L is formally self-adjoint.

(b) Define an extension $\bar{L}$ of L which is rigorously self-adjoint. You should justify your answer briefly, but a detailed proof is not required.

(c) If λ_1 is the smallest eigenvalue of L, prove that

$$0 < \lambda_1 \leq \frac{1}{2}.$$

5. Two-dimensional Minkowski space $\mathbf{M}$ is the vector space $\mathbf{R}^2$ with an inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + x_1 y_1,$$
$$\mathbf{x} = (x_0, x_1), \qquad \mathbf{y} = (y_0, y_1).$$

Is this space a Hilbert space? Why? Let V be a one-dimensional subspace of $\mathbf{M}$ spanned by the vector $\mathbf{e} = (\cos\theta, \sin\theta)$, where $0 \leq \theta \leq \pi/2$. Determine the orthogonal complement,

$$V^\perp = \{\mathbf{x} \in \mathbf{M} : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \text{for all } \mathbf{y} \in \mathbf{M}\},$$

and draw a picture. Is it always true that $\mathbf{M} = V \oplus V^\perp$?

6. Define the integral operator $K : C^\infty[0,+\infty) \to C^\infty[0,+\infty)$ by

$$Ku(x) = \int_0^x \frac{u(y)}{(x-y)^{1/2}}\,dy.$$

(a) Show that

$$K^2u = \pi \int_0^x u(y)\,dy.$$

Hint: Exchange the order of integration; you can assume that

$$\int_0^1 \frac{dt}{(1-t)^{1/2}t^{1/2}} = \pi.$$

(b) Use the result of (a) to deduce that the solution of the integral equation

$$\int_0^x \frac{u(y)}{(x-y)^{1/2}}\,dy = f(x)$$

is given by

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(y)}{(x-y)^{1/2}}\,dy.$$

7. Consider spatially periodic solutions $u(x,t)$ of the fourth order diffusion equation,

$$u_t = \nu u_{xxxx},$$
$$u(x,0) = f(x)$$
$$u(x+2\pi,t) = u(x,t).$$

Here ν is a nonzero constant and $x \in \mathbf{T}$ where $\mathbf{T}$ is the unit circle. Use Fourier series to find the solution of this problem. You can assume $f \in L^2(\mathbf{T})$. Discuss the existence and smoothness of the solution for $t > 0$. Consider both $\nu > 0$ and $\nu < 0$.

8. Find a leading order approximation as $\epsilon \to 0+$ of the solution $y(x;\epsilon)$ of the boundary value problem

$$\epsilon y'' - y' + 2xy = 0,$$
$$y(0;\epsilon) = 1,$$
$$y(1;\epsilon) = 0.$$

9. Consider a simple harmonic oscillator whose frequency ω varies slowly in time,

$$\ddot{y} + \omega^2(\epsilon t)y = 0.$$

Use the WKB method to obtain a leading order asymptotic solution as $\epsilon \to 0+$ which is valid for $t = O(1/\epsilon)$. Let $E(\epsilon t)$ be the energy of the oscillator and define the action by $S = E/\omega$. Show that S is asymptotically constant in time.