GGAM Preliminary Exam, Winter 97

1. Consider the second order equation

$$x'' = 1 - x + \epsilon x^2 - bx'$$

where $0 < \epsilon < 1/4$.
(a) Assume that there is no damping, that is, $b = 0$. Write down the equivalent first order system and show that it is a Hamiltonian system. Classify its fixed points and sketch the phase portrait.
(b) Keep the assumption that $b = 0$, and let $x(0) = 0$, $x'(0) = a$. Show that there exists $A(\epsilon)$ so that $\lim_{t\to\infty} x(t) = \infty$ if and only if $a > A(\epsilon)$. To show how fast $A(\epsilon)$ goes to ∞ as $\epsilon \to 0$, determine the power p for which $\lim_{\epsilon\to 0} \epsilon^p A(\epsilon)$ exists.
(c) Classify the fixed points if $b > 0$.

2. Consider the system

$$x' = -2x + 2(4 - x)y$$
$$y' = -y + (4 - y)x$$

Let $S = \{(x, y) : 0 \le x \le 4, 0 \le y \le 4\}$. Assume that $(x(0), y(0)) = (a, b) \in S$.
(a) Show that $(x(t), y(t)) \in S$ for all $t \ge 0$.
(b) Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. For which choices of (a, b) is $x(t)$ increasing for $t \ge 0$?
(c) Assume that (a, b) is not one of the fixed points. Compute

$$\lim_{t\to\infty} \frac{\ln|x(t) - y(t)|}{t}$$

3. Consider the system

$$x' = \alpha x + xy$$
$$y' = \alpha y - xy$$

For every $\alpha \in (-\infty, \infty)$, determine the set of fixed points and determine whether they are asymptotically stable, neutrally stable, or unstable.

4. Let $f \in C_0^\infty(-\infty, \infty)$, the space of real valued infinitely differentiable functions with compact support equipped with the inner product

$$(f, g) = \int_{-\infty}^{\infty} q(x)\ f(x) \cdot g(x)\ dx$$

where $q(x) > 0$ is C^∞. Consider the differentiable operator

$$L\ f = a(x)\ \frac{d^2\ f}{dx^2} + b(x)\ \frac{d\ f}{dx} + c(x)\ f(x)$$

where $a(x),\ b(x),\ c(x)$ are C^∞.
(i) Under what conditions is L self adjoint?
(ii) Under what conditions is L positive definite?
(iii) Under what conditions is L skew adjoint?

5. Let $f \in C_0(-\infty, \infty)$, a continuous function with compact support. Define a family of functions in C[0,1] equipped with the L_∞ norm by

$$F = \{f_\tau(t) = f(t + \tau) : \tau \in \mathbf{R}\}$$

Show that F is precompact.

6. Let X, Y be a Banach spaces and $L : X \to Y$ be a bounded and invertible operator. Let $L_n : X \to Y$ be a sequence of bounded and invertible operators which converge strongly to L, i.e., for all x, $\|(L - L_n)x\| \to 0$ as $n \to \infty$. Suppose there exists an M such that $\|L_n^{-1}\| \le M$ for all n. Let x and x_n be the solutions of $L\,x = f$ and $L_n\,x_n = f$. Show that $x_n \to x$.

7. Let $\{e_n\}$ be an orthonormal basis for a complex Hilbert space H. Let $\{\lambda_n\}$ be a collection of complex scalars. Formerly define a linear operator

$$L\,x = \sum_1^\infty \lambda_n\;(x, e_n)\;e_n$$

a) Show that L is well defined on a dense subset of H.
b) When is L continuous?
c) Suppose L is bounded, show it is normal.
d) When is L self adjoint?
e) When does L have a finite range?
f) Prove L is compact iff $|\lambda_n| \to 0$.