GGAM preliminary exam (Tuesday, 6 January 1998; 3 pages)

State carefully or indicate by name the theorems you use in your arguments.

Problem 1. Define three subsets of $C([0,1])$ as follows:

$$\begin{aligned} A_1 &= \{f \in C([0,1]) \mid |f(x)| \leq 1, \quad \text{for all } x \in [0,1]\} \\ A_2 &= \{f \in C^1([0,1]) \mid |f'(x)| \leq 2, \quad \text{for all } x \in [0,1]\} \\ A_3 &= \{f \in C([0,1]) \mid f \text{ is a polynomial function}\} \end{aligned}$$

and consider the following three intersections

$$B_1 = A_1 \cap A_2, \quad B_2 = A_1 \cap A_3, \quad B_3 = A_1 \cap A_2 \cap A_3$$

a) Is B_1 compact, precompact but not compact, or not precompact, considered as a subset of $(C([0,1]), \|\cdot\|_{\text{sup}})$?

b) Same question for B_2.

c) Same question for B_3.

Problem 2. Define two sequences of functions, (f_n) and (g_n), in $C([0,1])$ as follows:

$$\begin{aligned} f_n(x) &= (1 + \cos 2\pi x)^{1/n}, \quad n \geq 1 \\ g_n(x) &= (1 + \frac{1}{2}\cos 2\pi x)^{1/n}, \quad n \geq 1 \end{aligned}$$

a) What are the pointwise limits, f and g, of the sequences (f_n) and (g_n) respectively?

b) For each sequence, determine whether the convergence is uniform. Give proofs!

Problem 3. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\, dy + \frac{1}{2}\sin(f(x)) \qquad (*)$$

Prove that there exists a $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all $g \in C([0,1])$, (*) has a unique continuous solution.

Problem 4. Let X be a Banach space.

a) Prove that, for all $A \in \mathcal{B}(X)$, the following series converges in $\mathcal{B}(X)$ with the standard operator norm:

$$\exp A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

b) Prove that the map $\exp : \mathcal{B}(X) \to \mathcal{B}(X)$, as defined in **a)**, is continuous.

Problem 5. Let $f \in \mathcal{S}(\mathbb{R})$, the Schwarz space of test functions. Consider the following equation for $u \in L^2(\mathbb{R}, dx)$:

$$u''(x) + \sqrt{\pi} \int_{-\infty}^{+\infty} e^{-|x-y|} u(y)\, dy = f(x)$$

Find g such that the solution u of this equation can be expressed as

$$u(x) = \int_{-\infty}^{+\infty} g(x,y) f(y)\, dy$$

(Hint: use the Fourier Transform.)

Problem 6. For any pair of real numbers α and β define the function $F_{\alpha,\beta} : \mathbb{R}^3 \to \mathbb{R}$, by

$$F_{\alpha,\beta}(x) = \|x\|^\alpha (1 + \|x\|)^\beta$$

a) Let $p \in (1, +\infty)$. For which pairs of α and β does the functional

$$\phi_{\alpha,\beta}(f) := \int F_{\alpha,\beta}(x) f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

extend to a continuous linear functional on $L^p(\mathbb{R}^3, dx)$? Prove your answer.

b) The same question as in **a)** for the cases $p = 1$ and $p = \infty$.

Problem 7. Consider the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$

with initial condition $x_1(0) = a$, $x_2(0) = b$. Here, a, b, c are real numbers, and

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} .$$

a) Assume $c = 0$, and $(a, b) \neq (0, 0)$. Explain how the following limits depend on a and b:

$$\lim_{t\to\infty} x_2(t),$$

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)},$$

$$\lim_{t\to\infty} \frac{1}{t} \log x_2(t).$$

b) Assume that $c \neq 0$. Find all fixed points and discuss their stability.

Problem 8. A ball of unit mass is moving without friction in a potential well $V(x) = x^3 - 3x$. Recall that the equation of motion is $x'' = -V'(x)$, or $x' = y$, $y' = -V(x)$.

a) Show that $H(x, y) = \frac{1}{2}y^2 + V(x)$ is a conserved quantity for this sistem. Describe the behavior of trajectories by sketching the phase portrait.

b) Assume that $x'(0) = 0$, $x(0) = a$, so that the ball starts at rest at location a. Determine for which a is $x(t)$ a periodic function of t.

c) Consider now the forced equation $x'' = -V'(x) - \gamma$, where $\gamma > 0$. Determine the smallest γ for which periodic motion is no longer possible.

d) Finally, consider the damped equation $x'' = -V'(x) - bx'$, where $b > 0$. Show that $H(x, y)$ is non–increasing on every trajectory. Then show that $(0, 0)$ is a stable fixed point in this case.