Fall 2003 Mathematics Voluntary Assessment Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Algebra and Linear Algebra

Problem 1. A vector space V contains an n-element set with the following properties:
(a) It is not linearly independent, but contains an $(n-1)$-element linearly independent set;
(b) It does not span V, but is contained in an $(n+1)$-element spanning set.
Prove that $\dim V = n$.

Problem 2. Find all automorphisms of $\mathbb{Z}[x]$, the ring of polynomials over $\mathbb{Z}$.

Problem 3. Let G be a group and let H be a subgroup of G with finite index $n > 1$.
a. Show that the map $G \times G/H \to G/H$ defined by $(g, aH) \mapsto gaH$ gives an action of G on the space G/H of left cosets of H in G.
b. Show that if, in addition, G is finite and the order of G does not divide $n!$, then G is not simple.
c. Can a group of order $2^2 \cdot 3 \cdot 19^2$ be simple?

Problem 4. Let M and N be complex $n \times n$ matrices (possibly, singular). Prove that there exists a non-zero vector $v \in \mathbb{C}^n$ such that the vectors Mv and Nv are linearly dependent.

Problem 5. a. Construct an extension of degree 3 of $\mathbb{Q}$ which has an automorphism other than the identity.
b. Let F be a finite field. Show that the number of elements of F is p^r for some prime p and positive integer r.

2. Analysis

Problem 6. (a.) In each part, either give an example or explain briefly why no example can exist. (i) a sequence of real numbers a_n with $\liminf a_n = 0$ and $\limsup a_n = 5$
(ii) a bounded sequence with an unbounded subsequence
(iii) a convergent series $\sum_{n=1}^{\infty} a_n$ with $a_{n+1}/a_n \to 1$.
(iv)a convergent sequence (a_n) such that $1/a_n$ diverges.
(v) a convergent sequence (a_n) such that $(a_n)^2$ diverges.
(b) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that $\{x_n\}$ has a convergent subsequence.

Problem 7. (a) For a function $f : (a, b) \to R^1$, (a, b) an open interval, state briefly but precisely:
i. What is meant by the statement: $f(x)$ is continuous at $x_0 \in (a, b)$.
ii. What is meant by the statement: $f(x)$ is continuous on (a, b).
iii. What is meant by the statement: $f(x)$ is uniformly continuous on (a, b).
(b) Prove, directly from the definition, that the function $f(x) = 1/x$ is uniformly continuous on the interval $[1, \infty)$.

Problem 8. a. The function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } 0 \leq x < 1/2 \\ 0 & \text{if } 1/2 \leq x \leq 1 \end{array} \right\}$$

is not continuous. Demonstrate this using the definition of continuity involving open sets.
b. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a function and that $g^{-1}(-\infty, 0)$ is the set of rational numbers. Is it possible that g is continuous? Explain.

Problem 9. Consider the map $T : R^3 \to R^3$ given by

$$T(x_1, x_2, x_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3).$$

a. Compute the Jacobian matrix of T.
b. For which values of $x = (x_1, x_2, x_3)$ is the map locally invertible (i.e. there exists a neighborhood U of x and a neighborhood V of $T(x)$ such that $T : U \to V$ is 1-1 and onto with inverse map $T^{-1} : V \to U$).
c. Compute the Jacobian matrix of T^{-1} at $f(x)$ where it exists.