Voluntary Assessment Exam for 2004-05

Instructions: *explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Algebra and Linear Algebra

Problem 1. Let V be a nonzero finite-dimensional complex vector space, and let $f, g\colon V \to V$ be two linear maps. Prove that there exists a non-zero vector $v \in V$ such that the vectors $f(v), g(v)$ are collinear (that is, $\dim(\mathrm{Span}(f(v), g(v))) \leq 1$).

Warning: neither of f, g is assumed to be non-singular.

Problem 2. Prove that $\mathbb{Q}$ is not a nontrivial direct sum $A \oplus B$ of two subgroups.

Problem 3. True or false: Suppose that G is a finitely generated group and $H \subset G$ is a subgroup, so that H is isomorphic to G. Then $H = G$. Provide a proof or a counterexample.

Problem 4. Let A be a real 3×3 matrix auch that

$$\mathrm{Tr}\, A = \mathrm{Tr}(A^2) = \mathrm{Tr}(A^3) = 0.$$

Prove that $A^3 = 0$.

Problem 5. Give examples of each of the following

a) a finite nonabelian group

b) an infinite nonabelian group

c) a group that is not finitely generated

d) a group that is not solvable

Analysis

Problem 6. Define a sequence of real numbers (x_n) by

$$x_0 = 1, x_{n+1} := \frac{1}{2 + x_n}, n \geq 0.$$

Show that (x_n) converges and compute its limit.

Hint: Use the contraction principle.

Problem 7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function which for some $\lambda > 0$ satisfies

$$|f'(x)| \geq \lambda, \quad \forall x \in \mathbb{R}.$$

Prove that

a) For all $x, y \in \mathbb{R}$ the following inequality holds:

$$|f(x) - f(y)| \geq \lambda |x - y|.$$

b) $f : \mathbb{R} \to \mathbb{R}$ is one-to-one and onto.

Problem 8. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiualble for each n, so that

$$|f_n'(x)| \leq 1, \forall x \in \mathbb{R}, n \in \mathbb{N}.$$

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is such that for each $x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = g(x).$$

prove that g is continuous.

Problem 9. Suppose that X, Y are topological spaces, $f : X \to Y$ is a continuous mapping. Prove that if X is compact, then $f(X)$ is compact as well.

Problem 10. For each function $f : \mathbb{R} \to \mathbb{R}$ define the set $C(f)$ of points where f is continuous. Determine if the following true or false:

For each function f the set $C(f)$ is open in $\mathbb{R}$.

Give a proof or a justified counterexample.