

# Fall 2006: Voluntary Assessment Exam

## Instructions:

- (1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- (2) *Use separate sheets for the solution of each problem.*

## Algebra and Linear Algebra

**Problem 1.** Let  $A \in \mathbb{C}^{n \times n}$  be a unitary matrix.

- (a) Prove that all eigenvalues of  $A$  lie on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .
- (b) Determine all singular values of  $A$ .

**Problem 2.** Let  $v \in \mathbb{R}^n$  be a given vector with norm  $\|v\|_2 = 1$ . Consider the matrix  $P = I - vv^T$ , where  $I$  denotes the  $n \times n$  identity matrix.

- (a) Show that the matrix  $P$  is symmetric.
- (b) Show that  $P^2 = P$ .
- (c) Let  $b \in \mathbb{R}^n$  be any vector. Show that the two vectors  $b_1 = Pb$  and  $b_2 = (I - P)b$  are orthogonal.
- (d) Find a nonzero vector in the nullspace of  $P$ .

**Problem 3.** Let  $R$  be a commutative ring, and let  $I_n$  be an ideal of  $R$  for each  $n \geq 1$ . Suppose  $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ , and set

$$I := \bigcup_{n=1}^{\infty} I_n.$$

- (a) Prove that  $I$  is an ideal of  $R$ .
- (b) Prove that if  $I = R$ , then there exists an  $n \geq 1$  such that  $I_n = R$ .

**Problem 4.** Determine all ring automorphisms  $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$  that satisfy the additional condition  $\phi(r) = r$  for all  $r \in \mathbb{R}$ .

**Problem 5.** Let  $G$  be a group of order 75.

- (a) Show that  $G$  has a subgroup  $H$  of order 3. Must  $H$  be normal in  $G$ ?
- (b) What are the different possibilities for how many distinct elements of order 3 the group  $G$  has?

## Analysis

**Problem 6.** Define  $f_n, g_n : [0, 1] \rightarrow \mathbb{R}$  by

$$f_n(x) = \frac{nx^2}{1+n^2x^2} \quad \text{and} \quad g_n(x) = \frac{n^2x}{1+n^2x^2}.$$

Show that the sequences  $(f_n)$  and  $(g_n)$  converge pointwise on  $[0, 1]$ , and determine their pointwise limits. Determine (with proof) whether or not each sequence converges uniformly on  $[0, 1]$ .

**Problem 7.** Define  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is  $f$  Riemann integrable on  $[0, 1]$ ? Prove your answer.

**Problem 8.** Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \begin{cases} \frac{xy}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that the partial derivatives of  $f$  with respect to  $x$  and  $y$  exist at every point in  $\mathbb{R}^2$ . At which points do the directional derivatives of  $f$  exist in every direction? Where is  $f$  differentiable?

**Problem 9.** Let

$$\omega = f dx + dy \quad \text{and} \quad \lambda = g dx + dz$$

be one-forms in  $\mathbb{R}^3$ , where  $f(x, y, z)$  and  $g(x, y, z)$  are smooth functions.

- Calculate  $\omega \wedge \lambda$  and express your answer in standard form.
- Calculate  $d(\omega \wedge \lambda)$  and express your answer in standard form.

**Problem 10.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  denote the natural numbers, and define the functions  $d_1, d_2 : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  by

$$d_1(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right| \quad \text{and} \quad d_2(n, m) = |n - m|.$$

- Prove that  $d_1$  and  $d_2$  are metrics on  $\mathbb{N}$ .
- Determine whether or not  $\mathbb{N}$  is complete with respect to each of the metrics  $d_1$  and  $d_2$ .