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Abstract

We construct the Schubert basis of the torus-equivariant K-homology of the affine Grassmannian of a simple algebraic group G, using the K-theoretic NilHecke ring of Kostant and Kumar. This is the K-theoretic analogue of a construction of Peterson in equivariant homology. For the case where $G = \mathrm{SL}_n$, the K-homology of the affine Grassmannian is identified with a sub-Hopf algebra of the ring of symmetric functions. The Schubert basis is represented by inhomogeneous symmetric functions, called K-k-Schur functions, whose highest-degree term is a k-Schur function. The dual basis in K-cohomology is given by the affine stable Grothendieck polynomials, verifying a conjecture of Lam. In addition, we give a Pieri rule in K-homology. Many of our constructions have geometric interpretations by means of Kashiwara's thick affine flag manifold.

Contents

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1. Introduction

Let G be a simple simply connected complex algebraic group and $T \subset G$ the maximal torus. Let Gr_G denote the affine Grassmannian of G. The T-equivariant K-cohomology $K^T(\mathrm{Gr}_G)$ and K-homology $K_T(\mathrm{Gr}_G)$ are equipped with distinguished $K^T(\mathrm{pt})$-bases (denoted by $\{[\mathcal{O}_{X_w^I}]\}$ and $\{\xi_w\}$), called Schubert bases. Our first main result is a description of the K-homology $K_T(\mathrm{Gr}_G)$ as a subalgebra $\mathbb{L}$ of the affine K-NilHecke algebra of Kostant and Kumar [KK90]. This generalizes work of Peterson [Pet97] in homology. Our second main result is the identification, in the case where $G = \mathrm{SL}_n(\mathbb{C})$, of the Schubert bases of the non-equivariant K-homology $K_*(\mathrm{Gr}_G)$ and K-cohomology $K^*(\mathrm{Gr}_G)$ with explicit symmetric functions called *K-k-Schur functions* and *affine stable Grothendieck polynomials* [Lam06]. This generalizes work of Lam [Lam08] in (co)homology.

1.1 Kostant and Kumar's K-NilHecke ring

Let $\mathfrak{g}$ be a Kac–Moody algebra and let X be the flag variety of $\mathfrak{g}$. Kostant and Kumar [KK90] studied the equivariant K-theory $K^T(X)$ via a dual algebra $\mathbb{K}$ called the K-NilHecke ring. The ring $\mathbb{K}$ acts on $K^T(X)$ by Demazure divided difference operators and scalar multiplication by $K^T(\mathrm{pt})$. In particular, they used $\mathbb{K}$ to define a 'basis' $\{\psi^v_{KK}\}$ of $K^T(X)$ (elements of $K^T(X)$ are *infinite* $K^T(\mathrm{pt})$-linear combinations of the 'basis').

Kostant and Kumar use the ind-scheme X_{ind}, which is an inductive limit of finite-dimensional schemes. Because of this, classes in $K^T(X_{\mathrm{ind}})$ do not have an immediate geometric interpretation but, rather, are defined via duality in terms of geometric classes in $K_T(X_{\mathrm{ind}})$. We use instead the 'thick' flag variety X of Kashiwara [Kas89], which is an infinite-dimensional scheme. This allows us to interpret the K-NilHecke ring operations geometrically and to describe (in Theorem 3.2) the Schubert 'basis' of $K^T(X)$, representing coherent sheaves $\mathcal{O}_{X_w}$ of finite-codimensional Schubert varieties. Our basis is different from that of Kostant and Kumar. On the other hand, in our treatment the K-homology $K_T(X)$ is now defined via duality.

1.2 The affine Grassmannian and the small-torus GKM condition

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let $\mathfrak{g}_{\mathrm{af}}$ be the untwisted affine algebra. Instead of using the affine torus T_{af}, we use the torus $T \subset G$ of the finite-dimensional algebraic group and study the equivariant K-cohomology $K^T(X_{\mathrm{af}})$ and $K^T(\mathrm{Gr}_G)$ of the affine flag variety and affine Grassmannian. We use the *affine K-NilHecke ring for* $\mathfrak{g}$, still denoted by $\mathbb{K}$, rather than the slightly larger Kostant–Kumar K-NilHecke ring for $\mathfrak{g}_{\mathrm{af}}$. The corresponding affine NilHecke ring in cohomology was considered by Peterson [Pet97].

We describe (in Theorem 4.3) the image of $K^T(X_{\mathrm{af}})$ and $K^T(\mathrm{Gr}_G)$ in $\prod_{w\in W_{\mathrm{af}}} K^T(\mathrm{pt})$ under localization at the fixed points, where W_{af} denotes the affine Weyl group. This is the K-theoretic analogue of a result of Goresky *et al.* [GKM04] in homology. We call the corresponding condition the *small-torus GKM condition*. It is significantly more complicated than the usual condition for GKM spaces [GKM98], which would apply if we had used the larger torus T_{af}. This description gives an algebraic proof of the existence of a crucial 'wrong way' map $K^T(X_{\mathrm{af}}) \to K^T(\mathrm{Gr}_G)$, which corresponds in the topological category to $\Omega K \hookrightarrow LK \to LK/T_{\mathbb{R}}$ where $K \subset G$ is a maximal compact subgroup, $T_{\mathbb{R}} = T \cap K$, and ΩK and LK denote the spaces of based and unbased loops, respectively. The space of based loops ΩK is a topological model for the affine Grassmannian [PS86].

Another description of the K-homology of the affine Grassmannian is given by Bezrukavnikov *et al.* [BFM05], although the methods there do not appear to be particularly suited to the study of Schubert calculus.

1.3 The K-theoretic Peterson subalgebra and affine Fomin–Stanley subalgebra

We let $\mathbb{L} = Z_{\mathbb{K}}(R(T))$ denote the centralizer in $\mathbb{K}$ of the scalars $R(T) = K^T(\mathrm{pt})$ and call it the *K-Peterson subalgebra*. (This centralizer would be uninteresting if we had used T_{af} instead of T.) We generalize (in Theorem 5.3) the following result of Peterson [Pet97] (see also [Lam08]) in homology.

THEOREM. *There is a Hopf-isomorphism* $k : K_T(\mathrm{Gr}_G) \longrightarrow \mathbb{L}$.

The Hopf-structure of $K_T(\mathrm{Gr}_G)$ is derived from ΩK. We also give a description of the images $k(\xi_w)$ of the Schubert bases under this isomorphism (Theorem 5.4).

Next, we consider a subalgebra $\mathbb{L}_0 \subset \mathbb{K}_0$, called the *K-affine Fomin–Stanley subalgebra*, of the affine 0-Hecke algebra. We shall show that $\mathbb{L}_0$ is the evaluation of $\mathbb{L}$ at zero, and that it is a model for the non-equivariant homology $K_*(\mathrm{Gr}_G)$.

1.4 $G = \mathrm{SL}_n$ and Grothendieck polynomials for the affine Grassmannian

We now focus on the case where $G = \mathrm{SL}_n$. In [Lam06], the *affine stable Grothendieck polynomials* $G_w(x)$ were introduced, where $w \in W_{\mathrm{af}}$ is an affine permutation. The symmetric functions $G_w(x)$ lie in a completion $\hat{\Lambda}^{(n)}$ of a quotient of the ring of symmetric functions. A subset of the $\{G_w(x)\}$ form a basis of $\hat{\Lambda}^{(n)}$, and the dual basis elements $g_w(x)$, called *K-theoretic k-Schur functions*, form a basis of a subalgebra $\Lambda_{(n)}$ of the ring of symmetric functions.

The symmetric functions $G_w(x)$ are K-theoretic analogues of the affine Stanley symmetric functions in [Lam06] and, on the other hand, affine analogues of the stable Grothendieck polynomials in [Buc02, FK94]. The symmetric functions $g_w(x)$ are K-theoretic analogues of the k-Schur functions $s_w(x)$ (see [LLM03, LM05, LM07]) and, on the other hand, affine (or k-) analogues of the dual stable Grothendieck polynomials [LP07, Len00].

Using the technology of the K-affine Fomin–Stanley subalgebra, we confirm a conjecture of Lam [Lam06] by proving the following result (see Theorem 7.17).

Theorem. *There are Hopf-isomorphisms $K_*(\mathrm{Gr}_G) \cong \Lambda_{(n)}$ and $K^*(\mathrm{Gr}_G) \cong \hat{\Lambda}^{(n)}$ that identify the homology Schubert basis with the K-k-Schur functions $g_w(x)$ and the cohomology Schubert basis with the affine stable Grothendieck polynomials $G_w(x)$.*

This generalizes the main result of [Lam08], and the general idea of the proof is the same.

We also obtain a Pieri rule (Corollary 7.6) for $K_*(\mathrm{Gr}_G)$. We give in Theorem 7.19 a geometric interpretation of $G_w(x)$ for any $w \in W_{\mathrm{af}}$ as a pullback of a Schubert class from the affine flag variety to the affine Grassmannian. We conjecture that the symmetric functions $G_w(x)$ and $g_w(x)$ satisfy many positivity properties (Conjectures 7.20 and 7.21).

1.5 Related work

Morse [Mor] gives a combinatorial definition of the affine stable Grothendieck polynomials $G_w(x)$ in terms of affine set-valued tableaux and also proves the Pieri rule for G_w. The original k-Schur functions $s_w(x;t)$ in [LLM03, LM05], which arose in the study of Macdonald polynomials, involve a parameter t. It appears that a t-analogue $g_w(x;t)$ of $g_w(x)$ exists, defined in a similar manner to [LLMS, Conjecture 9.11]. The connection between $g_w(x;t)$ and Macdonald theory is explored in [BM].

Kashiwara and Shimozono [KS09] constructed polynomials, called *affine Grothendieck polynomials*, which represent Schubert classes in the K-theory of the affine flag manifold. It is unclear how affine Grothendieck polynomials compare with our symmetric functions.

1.6 Organization of the paper

In § 2 we review the constructions of the K-NilHecke ring $\mathbb{K}$ and define our function 'basis' $\{\psi^v\}$. In § 3 we introduce Kashiwara's geometry of 'thick' Kac–Moody flag varieties X and the corresponding equivariant K-cohomologies; we show how $\mathbb{K}$ corresponds to the geometry of X. Section 4 is devoted to equivariant K-theory for affine flags and affine Grassmannians with the small torus T acting by means of the level-zero action of the affine Weyl group. In § 5 we introduce the affine K-NilHecke ring and the K-Peterson subalgebra $\mathbb{L}$, and we prove that the latter is isomorphic to $K_T(\mathrm{Gr}_G)$. In § 6 we study the K-affine Fomin–Stanley algebra. In § 7 we restrict to $G=\mathrm{SL}_n$ and describe explicitly the Hopf-algebra isomorphisms between $K_*(\mathrm{Gr}_G)$ and $K^*(\mathrm{Gr}_G)$ and symmetric functions.

A review of the cohomological NilHecke ring of Kostant and Kumar and the affine NilHecke algebra $\mathbb{A}$ as well as some tables of the symmetric functions g_w and G_w are provided in Appendix A.

2. The Kostant–Kumar K-NilHecke ring

One of the themes of [KK90] is that the Schubert calculus of the torus-equivariant K-theory $K^T(X)$ of a Kac–Moody flag manifold X is encoded by the K-NilHecke ring $\mathbb{K}$, which acts on $K^T(X)$ as *Demazure operators*. We review the constructions of [KK90] but use a different 'basis' for $K^T(X)$, namely, the classes of equivariant structure sheaves of finite-codimensional Schubert varieties in the thick flag manifold of [Kas89].

For a statement S, we write $\chi(S)=1$ if S is true and $\chi(S)=0$ if S is false.

2.1 Kac–Moody algebras

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the following data: a Dynkin node set I, a symmetrizable generalized Cartan matrix $(a_{ij})_{i,j\in I}$, a free $\mathbb{Z}$-module P, linearly independent simple roots $\{\alpha_i \mid i\in I\}\subset P$, and the dual lattice $P^*=\mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z})$ with simple coroots $\{\alpha_i^\vee \mid i\in I\}\subset P^*$ such that $\langle \alpha_i^\vee,\alpha_j\rangle = a_{ij}$ where $\langle\cdot,\cdot\rangle : P^*\times P\to\mathbb{Z}$ is the pairing, with the additional property that there exist fundamental weights $\{\Lambda_i \mid i\in I\}\subset P$ satisfying $\langle\alpha_i^\vee,\Lambda_j\rangle=\delta_{ij}$. Let $Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i\subset P$ be the root lattice and $Q^\vee=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee\subset P^*$ the coroot lattice. Let $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{t}\oplus\mathfrak{n}_-$ be the triangular decomposition, with $\mathfrak{t}\supset P^*\otimes_{\mathbb{Z}}\mathbb{C}$. Let Φ be the set of roots and $\Phi^\pm$ the sets of positive and negative roots, and let $\mathfrak{g}=\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ be the root space decomposition. Let $W\subset\mathrm{Aut}(\mathfrak{t}^*)$ be the Weyl group, with involutive generators r_i for $i\in I$ defined by $r_i\cdot\lambda=\lambda-\langle\alpha_i^\vee,\lambda\rangle\alpha_i$. For $i,j\in I$ with $i\neq j$, let m_{ij} be $2,3,4,6$ or ∞ according to whether $a_{ij}a_{ji}$ is $0,1,2,3$ or at least 4. Then W has involutive generators $\{r_i\mid i\in I\}$ which satisfy the braid relations $(r_ir_j)^{m_{ij}}=\mathrm{id}$. Let $\Phi^{\mathrm{re}}=\{w\alpha_i\mid w\in W, i\in I\}\subset Q$ be the set of real roots, and for $\alpha=w\alpha_i$ let $r_\alpha=wr_iw^{-1}$ be the associated reflection and $\alpha^\vee=w\alpha_i^\vee$ the associated coroot. Let $\Phi^{+\mathrm{re}}=\Phi^{\mathrm{re}}\cap\Phi^+$ be the set of positive real roots.

2.2 The rational form

Let T be the algebraic torus with character group P. The Weyl group W acts on P and therefore on $R(T)$ and $Q(T)=\mathrm{Frac}(R(T))$, where

$$R(T)\cong\mathbb{Z}[P]=\bigoplus_{\lambda\in P}\mathbb{Z}e^\lambda$$

is the Grothendieck group of the category of finite-dimensional T-modules. Here e^λ, for $\lambda\in P$, is the class of the one-dimensional T-module with character λ.

Let $\mathbb{K}_{Q(T)}$ be the smash product of the group algebra $\mathbb{Q}[W]$ and $Q(T)$, defined by $\mathbb{K}_{Q(T)}=Q(T)\otimes_{\mathbb{Q}}\mathbb{Q}[W]$ with multiplication

$$(q\otimes w)(p\otimes v)=q(w\cdot p)\otimes wv$$

for $p, q \in Q(T)$ and $v, w \in W$. We write qw instead of $q \otimes w$. For $i \in I$, define the Demazure operator [Dem74] $y_i \in \mathbb{K}_{Q(T)}$ by

$$y_i = (1 - e^{-\alpha_i})^{-1}(1 - e^{-\alpha_i} r_i).$$

The y_i are idempotent and satisfy the braid relations

$$y_i^2 = y_i \quad \text{and} \quad \underbrace{y_i y_j \cdots}_{m_{ij} \text{ times}} = \underbrace{y_j y_i \cdots}_{m_{ij} \text{ times}}.$$

Define the elements $T_i \in \mathbb{K}_{Q(T)}$ by

$$T_i = y_i - 1 = (1 - e^{\alpha_i})^{-1}(r_i - 1). \tag{2.1}$$

We have

$$r_i = 1 + (1 - e^{\alpha_i})T_i. \tag{2.2}$$

The T_i satisfy

$$T_i^2 = -T_i \quad \text{and} \quad \underbrace{T_i T_j \cdots}_{m_{ij} \text{ times}} = \underbrace{T_j T_i \cdots}_{m_{ij} \text{ times}}. \tag{2.3}$$

Let $T_w = T_{i_1} T_{i_2} \cdots T_{i_N} \in \mathbb{K}_{Q(T)}$, where $w = r_{i_1} r_{i_2} \cdots r_{i_N}$ is a reduced decomposition; it is well-defined by (2.3). One can easily verify that

$$T_i T_w = \begin{cases} T_{r_i w} & \text{if } r_i w > w, \\ -T_w & \text{if } r_i w < w, \end{cases} \quad \text{and} \quad T_w T_i = \begin{cases} T_{w r_i} & \text{if } w r_i > w, \\ -T_w & \text{if } w r_i < w, \end{cases}$$

where $<$ denotes the Bruhat order on W. For $\alpha \in \Phi^{+\mathrm{re}}$, define $T_\alpha = (1 - e^{\alpha})^{-1}(r_\alpha - 1)$. Let $w \in W$ and $i \in I$ be such that $\alpha = w\alpha_i$. Then

$$T_\alpha = w T_i w^{-1}. \tag{2.4}$$

Note that $\mathbb{K}_{Q(T)}$ acts naturally on $Q(T)$; in particular, one has

$$T_i \cdot (qq') = (T_i \cdot q)q' + (r_i \cdot q) T_i \cdot q' \quad \text{for } q, q \in Q(T). \tag{2.5}$$

Therefore, in $\mathbb{K}_{Q(T)}$ we have

$$T_i\, q = (T_i \cdot q) + (r_i \cdot q) T_i \quad \text{for } q \in Q(T). \tag{2.6}$$

2.3 The 0-Hecke ring and integral form

The 0-Hecke ring $\mathbb{K}_0$ is the subring of $\mathbb{K}_{Q(T)}$ generated by the T_i. It can also be defined by generators $\{T_i \mid i \in I\}$ and the relations (2.3). We have $\mathbb{K}_0 = \bigoplus_{w \in W} \mathbb{Z} T_w$.

Lemma 2.1. *$\mathbb{K}_0$ acts on $R(T)$.*

Proof. We have that $\mathbb{K}_0$ acts on $Q(T)$; also, each T_i preserves $R(T)$ by (2.5) and the following formulae for $\lambda \in P$:

$$T_i \cdot e^\lambda = \begin{cases} e^\lambda (1 + e^{\alpha_i} + \cdots + e^{(\langle \alpha_i^\vee, \lambda \rangle - 1)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle > 0, \\ 0 & \text{if } \langle \alpha_i^\vee, \lambda \rangle = 0, \\ -e^\lambda (1 + e^{\alpha_i} + \cdots + e^{(-\langle \alpha_i^\vee, \lambda \rangle - 1)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle < 0. \end{cases}$$

The assertion follows. ∎

Define the *K-NilHecke ring* $\mathbb{K}$ to be the subring of $\mathbb{K}_{Q(T)}$ generated by $\mathbb{K}_0$ and $R(T)$. We have $\mathbb{K}_{Q(T)} \cong Q(T) \otimes_{R(T)} \mathbb{K}$. By (2.6),

$$\mathbb{K} = \bigoplus_{w \in W} R(T) T_w. \tag{2.7}$$

2.4 Duality and function 'basis'

Let $\mathrm{Fun}(W, Q(T))$ be the right $Q(T)$-algebra of functions from W to $Q(T)$ under pointwise multiplication and scalar multiplication $(\psi \cdot q)(w) = q\psi(w)$ for $q \in Q(T)$, $\psi \in \mathrm{Fun}(W, Q(T))$ and $w \in W$. By linearity, we identify $\mathrm{Fun}(W, Q(T))$ with left $Q(T)$-linear maps $\mathbb{K}_{Q(T)} \to Q(T)$ such that

$$\psi\left(\sum_{w \in W} a_w w\right) = \sum_{w \in W} a_w \psi(w).$$

Note that $\mathrm{Fun}(W, Q(T))$ is a $\mathbb{K}_{Q(T)}$–$Q(T)$-bimodule via

$$(a \cdot \psi \cdot q)(b) = \psi(qba) = q\psi(ba) \tag{2.8}$$

for $\psi \in \mathrm{Fun}(W, Q(T))$, $q \in Q(T)$ and $a, b \in \mathbb{K}_{Q(T)}$.

Evaluation gives a perfect pairing $\langle \cdot, \cdot \rangle : \mathbb{K}_{Q(T)} \times \mathrm{Fun}(W, Q(T)) \longrightarrow Q(T)$ defined by

$$\langle a, \psi \rangle = \psi(a),$$

which is $Q(T)$-bilinear in the sense that

$$\langle qa, \psi \rangle = q\langle a, \psi \rangle = \langle a, \psi \cdot q \rangle.$$

Define the subring $\Psi \subset \mathrm{Fun}(W, Q(T))$ by

$$\begin{aligned} \Psi &= \{\psi \in \mathrm{Fun}(W, Q(T)) \mid \psi(\mathbb{K}) \subset R(T)\} \\ &= \{\psi \in \mathrm{Fun}(W, Q(T)) \mid \psi(T_w) \in R(T) \text{ for all } w \in W\}. \end{aligned} \tag{2.9}$$

Clearly, Ψ is a $\mathbb{K}$–$R(T)$-bimodule. By (2.7), for $v \in W$ there are unique elements $\psi^v \in \Psi$ such that

$$\psi^v(T_w) = \delta_{v,w} \tag{2.10}$$

for all $w \in W$. We have $\Psi = \prod_{v \in W} R(T)\psi^v$.

Remark 2.1. In § 3 we show that $\psi^v(w)$ is the restriction of the equivariant structure sheaf $[\mathcal{O}_{X_v}]$ of the finite-codimensional Schubert variety $X_v \subset X$ of the thick Kac–Moody flag manifold X to the T-fixed point w. See Appendix A.2 for the relationship between our functions ψ^v and those of [KK90].

Letting $w = \mathrm{id}$, we have

$$\delta_{v,\mathrm{id}} = \psi^v(T_{\mathrm{id}}) = \psi^v(\mathrm{id}). \tag{2.11}$$

LEMMA 2.2. *For $v \in W$ and $i \in I$,*

$$y_i \cdot \psi^v = \begin{cases} \psi^{vr_i} & \text{if } vr_i < v, \\ \psi^v & \text{if } vr_i > v. \end{cases}$$

Proof. For $w \in W$ we have $T_w y_i = T_w(1+T_i) = \chi(wr_i > w)(T_w + T_{wr_i})$. Therefore, by (2.8),

$$\begin{aligned}\langle T_w, y_i \cdot \psi^v \rangle &= \langle T_w y_i, \psi^v \rangle \\ &= \chi(wr_i > w)\langle T_w + T_{wr_i}, \psi^v \rangle \\ &= \chi(wr_i > w)(\delta_{v,w} + \delta_{v,wr_i}) \\ &= \chi(vr_i > v)\delta_{v,w} + \chi(v > vr_i)\delta_{vr_i,w},\end{aligned}$$

from which the lemma follows. □

Remark 2.2. From (2.11) and Lemma 2.2 we obtain the following 'right-hand' recurrence for $\psi^v(w)$.

(i) If $w = \mathrm{id}$, then $\psi^v(\mathrm{id}) = \delta_{v,\mathrm{id}}$.

(ii) Otherwise, let $i \in I$ be such that $wr_i < w$. Then

$$\psi^v(w) = \begin{cases} \psi^v(wr_i) & \text{if } v < vr_i, \\ (1 - e^{-w(\alpha_i)})\psi^{vr_i}(w) + e^{-w(\alpha_i)}\psi^v(wr_i) & \text{if } vr_i < v. \end{cases} \tag{2.12}$$

This rewrites $\psi^v(w)$ in terms of $\psi^{v'}(w')$ for (v', w') such that either $w' < w$ or both $w' = w$ and $v' < v$.

LEMMA 2.3. *We have* $\psi^v(w) = 0$ *unless* $v \leqslant w$.

Proof. The statement is true for $w = \mathrm{id}$ by (2.11). Otherwise, let $i \in I$ be such that $wr_i < w$ and suppose that $v \nleqslant w$. Then $v \nleqslant wr_i$ and $vr_i \nleqslant w$ (see [Hum90]). The assertion is proved by induction using (2.12). □

The next result follows from the definitions.

PROPOSITION 2.4. *For all* $v, w \in W$, *we have* $w = \sum_{v \in W} \psi^v(w) T_v$.

Remark 2.3. Proposition 2.4 leads to a 'left-hand' recurrence for $\psi^v(w)$ as follows.

(i) For $w = \mathrm{id}$ we have (2.11).

(ii) Otherwise, let $i \in I$ be such that $r_i w < w$. By induction on length, we obtain

$$\begin{aligned} w &= r_i(r_i w) \\ &= (1 + (1 - e^{\alpha_i})T_i)\left(\sum_u \psi^u(r_i w) T_u\right) \\ &= \sum_u \psi^u(r_i w) T_u + (1 - e^{\alpha_i}) \sum_u T_i \psi^u(r_i w) T_u \\ &= \sum_u \psi^u(r_i w) T_u + (1 - e^{\alpha_i}) \sum_u ((T_i \cdot \psi^u(r_i w)) T_u + (r_i \cdot \psi^u(r_i w)) T_i T_u). \end{aligned}$$

Taking the coefficient of T_v, we see that

$$\psi^v(w) = \psi^v(r_i w) + (1 - e^{\alpha_i})((T_i \cdot \psi^v(r_i w)) + \chi(r_i v < v) r_i \cdot (\psi^{r_i v}(r_i w) - \psi^v(r_i w))).$$

Therefore, for $r_i w < w$ we have

$$\psi^v(w) = \begin{cases} r_i \cdot \psi^v(r_i w) & \text{if } r_i v > v, \\ e^{\alpha_i}\, r_i \cdot \psi^v(r_i w) + (1 - e^{\alpha_i})\, r_i \cdot \psi^{r_i v}(r_i w) & \text{if } r_i v < v. \end{cases}$$

Define the inversion set of $v \in W$ by

$$\mathrm{Inv}(v) = \{\alpha \in \Phi^{+\mathrm{re}} \mid r_\alpha v < v\}.$$

LEMMA 2.5. *For all $v \in W$, we have $\psi^v(v) = \prod_{\alpha \in \mathrm{Inv}(v)} (1 - e^\alpha)$.*

Proof. This follows directly from Remark 2.3 and Lemma 2.3. □

Remark 2.4. We have $\psi^v(w) = \eta(w \cdot \psi^{v^{-1}}(w^{-1}))$, where $\eta : \mathbb{Z}[P] \to \mathbb{Z}[P]$ is given by $\eta(e^\lambda) = e^{-\lambda}$.

2.5 The GKM condition

We recall the K-theoretic Goresky–Kottwitz–Macpherson (GKM) condition as a criterion for membership in Ψ. This condition and the associated geometry is discussed in § 3.3.

PROPOSITION 2.6. *Ψ is the set of $\psi \in \mathrm{Fun}(W, Q(T))$ such that*

$$\psi(r_\alpha w) - \psi(w) \in (1 - e^\alpha) R(T) \quad \text{for all } \alpha \in \Phi^{+\mathrm{re}} \text{ and } w \in W. \tag{2.13}$$

Proof. Let $\beta = w^{-1}\alpha$ and $\psi \in \Psi$. Then $r_\alpha w = w r_\beta$ and

$$\begin{aligned} \frac{\psi(r_\alpha w) - \psi(w)}{1 - e^\alpha} &= \psi((1 - e^\alpha)^{-1}(w r_\beta - w)) \\ &= \psi(w T_\beta), \end{aligned}$$

which is in $R(T)$ since $wT_\beta \in \mathbb{K}$, by using (2.4) and (2.2).

For the converse, let $\psi \in \mathrm{Fun}(W, Q(T))$ satisfy (2.13) and suppose that $\psi \neq 0$. Let $v \in \mathrm{Supp}(\psi)$ be a minimal element. For every $\alpha \in \Phi^{+\mathrm{re}}$ such that $r_\alpha v < v$, we have $\psi(v) \in (1 - e^\alpha) R(T)$ by (2.13), Lemma 2.3 and the minimality of v. Since the factors $(1 - e^\alpha)$ are relatively prime by [Kac90, Proposition 6.3], $\psi(v) \in \psi^v(v) R(T)$ by Lemma 2.5. Then $\psi' \in \Psi$, where $\psi'(w) = \psi(w) - (\psi(v)/\psi^v(v))\psi^v(w)$ for $w \in W$. Moreover, $v \notin \mathrm{Supp}(\psi')$ and $\mathrm{Supp}(\psi') \setminus \mathrm{Supp}(\psi)$ consists of elements strictly greater than v. Repeating the argument for ψ' and so on, we see that ψ is in $\prod_{v \in W} R(T) \psi^v$. □

2.6 Structure constants and coproduct

The proof of the following result is straightforward but lengthy.

PROPOSITION 2.7. *Let M and N be left $\mathbb{K}$-modules. Define*

$$M \otimes_{R(T)} N = (M \otimes_{\mathbb{Z}} N) / \langle qm \otimes n - m \otimes qn \mid q \in R(T), m \in M, n \in N \rangle.$$

Then $\mathbb{K}$ acts on $M \otimes_{R(T)} N$ by

$$\begin{aligned} q \cdot (m \otimes n) &= qm \otimes n, \\ T_i \cdot (m \otimes n) &= T_i \cdot m \otimes n + m \otimes T_i \cdot n + (1 - e^{\alpha_i}) T_i \cdot m \otimes T_i \cdot n. \end{aligned}$$

Under this action we have

$$w \cdot (m \otimes n) = wm \otimes wn. \tag{2.14}$$

Consider the case where $M = N = \mathbb{K}$. By Proposition 2.7 there is a left $R(T)$-module homomorphism $\Delta : \mathbb{K} \to \mathbb{K} \otimes_{R(T)} \mathbb{K}$ defined by $\Delta(a) = a \cdot (1 \otimes 1)$. It satisfies

$$\Delta(q) = q \otimes 1 \quad \text{for } q \in R(T), \tag{2.15}$$

$$\Delta(T_i) = 1 \otimes T_i + T_i \otimes 1 + (1 - e^{\alpha_i}) T_i \otimes T_i \quad \text{for } i \in I. \tag{2.16}$$

Let $a \in \mathbb{K}$ and $\Delta(a) = \sum_{v,w} a_{v,w} T_v \otimes T_w$ with $a_{v,w} \in R(T)$. It follows from Proposition 2.7 that the action of a on $M \otimes_{R(T)} N$ can be computed in the following simple 'componentwise' fashion: $a \cdot (m \otimes n) = \sum_{v,w} a_{v,w} T_v m \otimes T_w n$. In particular, if $b \in \mathbb{K}$ and $\Delta(b) = \sum_{v',w'} b_{v',w'} T_{v'} \otimes T_{w'}$, then

$$\Delta(ab) = \Delta(a) \cdot \Delta(b) := \sum_{v,w,v',w'} a_{v,w} b_{v',w'} T_v T_{v'} \otimes T_w T_{w'}. \tag{2.17}$$

Remark 2.5. The naive componentwise product is ill-defined on all of $\mathbb{K} \otimes_{R(T)} \mathbb{K}$, for if it were well-defined, then $(T_i \otimes 1)(q \otimes 1) = (T_i \otimes 1)(1 \otimes q)$ or, equivalently, $T_i q \otimes 1 = T_i \otimes q = q(T_i \otimes 1) = qT_i \otimes 1$, which is false for $q = e^{\alpha_i}$.

There is a left $R(T)$-bilinear pairing $\langle \cdot, \cdot \rangle : (\mathbb{K} \otimes_{R(T)} \mathbb{K}) \times (\Psi \otimes_{R(T)} \Psi) \to R(T)$ given by

$$\langle a \otimes b, \phi \otimes \psi \rangle = \langle a, \phi \rangle \langle b, \psi \rangle.$$

Lemma 2.8. *For all $a \in \mathbb{K}$ and $\phi, \psi \in \Psi$, we have $\langle a, \phi\psi \rangle = \langle \Delta(a), \phi \otimes \psi \rangle$.*

Proof. First, extend the definitions in the obvious manner to $\mathbb{K}_{Q(T)}$ and $\mathrm{Fun}(W, Q(T))$. Using left $Q(T)$-linearity, we may then take $a = w$. Then $\langle \Delta(w), \phi \otimes \psi \rangle = \langle w \otimes w, \phi \otimes \psi \rangle = \phi(w)\psi(w) = \langle w, \phi\psi \rangle$. □

Define the structure 'constants' $c_w^{uv} \in R(T)$ by $\psi^u \psi^v = \sum_{w \in W} c_w^{uv} \psi^w$. The structure constants of Ψ are recovered by the map Δ.

Proposition 2.9. *We have $\Delta(T_w) = \sum_{u,v} c_w^{uv}\, T_u \otimes T_v$ for all $w \in W$.*

Proof. This follows from Lemma 2.8 and the fact that $\psi^u(T_v) = \delta_{uv}$. □

2.7 Explicit localization formulae

For the sake of completeness, we give an explicit formula for the values $\psi^v(w)$. It is a variant of a formula due independently to Graham [Gra02] and Willems [Wil04]. Let $\varepsilon : \mathbb{K}_{Q(T)} \to Q(T)$ be the left $Q(T)$-module homomorphism defined by $\varepsilon(w) = 1$ for all $w \in W$.

Proposition 2.10. *Let $v, w \in W$ and let $w = r_{i_1} r_{i_2} \cdots r_{i_N}$ be any reduced decomposition of w. For $b_1 b_2 \cdots b_N \in \{0,1\}^N$, let $|b| = \sum_{i=1}^N b_i$. Then*

$$\psi^v(w) = \varepsilon \sum_{b \in B(i_\bullet, v)} (-1)^{\ell(w)-|b|} \prod_{k=1}^{N} \begin{cases} (1 - e^{\alpha_{i_k}}) r_{i_k} & \text{if } b_k = 1, \\ r_{i_k} & \text{if } b_k = 0, \end{cases} \tag{2.18}$$

where the sum runs over

$$B(i_\bullet, v) = \left\{ b = (b_1, b_2, \ldots, b_N) \in \{0,1\}^N \;\middle|\; \prod_{\substack{k \\ b_k = 1}} T_{i_k} = \pm T_v \right\}. \tag{2.19}$$

Formula (2.18) is the K-theoretic analogue of the formula [AJS94, Bil99, (1.2)] for the restriction of a T-equivariant Schubert cohomology class $[X_v]$ to a T-fixed point w.

Example 2.6. Let $G = \mathrm{SL}_3$, $v = r_1$ and $w = r_1r_2r_1$. Then there are three possible binary words b, namely $(1,0,0)$, $(0,0,1)$ and $(1,0,1)$, yielding

$$\begin{aligned}\psi^v(w) &= \varepsilon((1-e^{\alpha_1})r_1r_2r_1 + r_1r_2(1-e^{\alpha_1})r_1 - (1-e^{\alpha_1})r_1r_2(1-e^{\alpha_1})r_1) \\ &= (1-e^{\alpha_1}) + (1-e^{r_1r_2\alpha_1}) - (1-e^{\alpha_1})(1-e^{r_1r_2\alpha_1}) \\ &= (1-e^{\alpha_1}) + (1-e^{\alpha_2}) - (1-e^{\alpha_1})(1-e^{\alpha_2}) \\ &= 1-e^{\alpha_1+\alpha_2}.\end{aligned}$$

Using the reduced decomposition $w = r_2r_1r_2$ instead, there is only one summand $b = (0,1,0)$, and we obtain

$$\psi^v(w) = \varepsilon(r_2(1-e^{\alpha_1})r_1r_2) = 1-e^{r_2\alpha_1} = 1-e^{\alpha_1+\alpha_2}.$$

Proof of Proposition 2.10. Let the right-hand side of (2.18) be denoted by $\phi^v(w)$. We prove that $\psi^v(w) = \phi^v(w)$ by induction on w and then on v. Let $i = i_N$. If $v < vr_i$, then in any summand b we have $b_N = 0$, so that $\phi^v(w) = \phi^v(wr_i) = \psi^v(wr_i) = \psi^v(w)$ by induction on the length $\ell(w)$ of w and (2.12). Otherwise, let $v > vr_i$. The part of $\phi^v(w)$ with $b_N = 0$ is given by $\phi^v(wr_i) = \psi^v(wr_i)$. The rest of $\phi^v(w)$ consists of summands with $b_N = 1$. Consider the left-to-right product $\pm T_u$ of T_{i_k} for which $b_k = 1$ except that the term $b_N = 1$ is omitted. For the b such that $u = v$, the last factor $T_{i_N} = T_i$ produces an additional negative sign, and we obtain $-(1-e^{wr_i(\alpha_i)})\phi^v(wr_i) = (e^{-w(\alpha_i)}-1)\psi^v(wr_i)$ because the product $r_{i_1}\cdots r_{i_{N-1}}$ of the reflections is wr_i. For the b with $u = vr_i$, we obtain that $(wr_i\cdot(1-e^{\alpha_i}))\phi^{vr_i}(w) = (1-e^{-w(\alpha_i)})\psi^{vr_i}(w)$. In total, we obtain the right-hand side of (2.12), which equals $\psi^v(w)$. □

3. Equivariant K-cohomology of Kac–Moody flag manifolds

Kostant and Kumar [KK90] use the 'thin' Kac–Moody flag manifold X_{ind}, which is an ind-scheme with finite-dimensional Schubert varieties [Kum02]. In contrast, we employ the larger 'thick' Kac–Moody flag manifold X [Kas89], which is a scheme of infinite type with finite-codimensional Schubert varieties. Using the thick Kac–Moody flag manifold, we give natural geometric interpretations to the constructions in the K-NilHecke ring.

3.1 Kac–Moody thick flags

For the following discussion see [Kas89]. Let T be the algebraic torus with character group P, $U_\pm$ the group scheme with $\mathrm{Lie}(U_\pm) = \mathfrak{n}_\pm$, and $B_\pm$ the Borel subgroups with $\mathrm{Lie}(B_\pm) = \mathfrak{t}\oplus\mathfrak{n}_\pm$. For $i \in I$, let $P_i^\pm$ be the parabolic group with $\mathrm{Lie}(P_i^\pm) = \mathrm{Lie}(B_\pm)\oplus\mathfrak{g}_{\pm\alpha_i}$. These groups are all contained in an affine scheme G_∞ of infinite type that contains a canonical 'identity' point e. Let $G \subset G_\infty$ be the open subset defined by $G = \cup P_{i_1}\cdots P_{i_m} e P_{j_1}^- P_{j_2}^- \cdots P_{j_m}^-$. It is not a group but admits a free left action by each P_i and a free right action by each P_j^-.

Given the above, $X = G/B_-$ is then the thick Kac–Moody flag manifold; it is a scheme of infinite type over $\mathbb{C}$. For each subset $J \subset I$, let $P_J^- \subset G$ be the group generated by B_- and P_i^- for $i \in J$.

Write $X^J = G/P_J^-$. Let $W_J \subset W$ be the subgroup generated by r_i for $i \in J$, and let W^J be the set of minimal-length coset representatives in W/W_J. For $w \in W^J$, let $\mathring{X}_w^J = BwP_J^-/P_J^-$ where $B = B_+$; it is locally closed in X^J. We have the B-orbit decomposition

$$X^J = \bigsqcup_{w\in W^J} \mathring{X}_w^J.$$

Let $X_w^J = \overline{\mathring{X}_w^J}$ be the Schubert variety. It has codimension $\ell(w)$ in X^J and coherent structure sheaf $\mathcal{O}_{X_w^J}$. We have

$$X_w^J = \bigsqcup_{\substack{v \in W^J \\ v \geqslant w}} \mathring{X}_v^J. \tag{3.1}$$

Let S be a finite Bruhat order ideal of W^J (a finite subset $S \subset W^J$ such that if $v \in S$, $u \in W^J$ and $u \leqslant v$, then $u \in S$). Let $\Omega_S^J = \bigsqcup_{w \in S} \mathring{X}_w^J = \bigcup_{w \in S} wBP_J^-/P_J^-$ be a B-stable finite union of translations of the big cell $\mathring{X}^J := BP_J^-/P_J^-$, which is open in X^J. The big cell is an affine space of countable dimension (finite if $\mathfrak{g}$ is finite-dimensional): $\mathring{X}^J \cong \operatorname{Spec}(\mathbb{C}[x_1, x_2, \dots])$.

3.2 Equivariant K-cohomology

Denote by $\operatorname{Coh}^T(\Omega_S^J)$ the category of coherent T-equivariant $\mathcal{O}_{\Omega_S^J}$-modules, and let $K^T(\Omega_S^J)$ be the Grothendieck group of $\operatorname{Coh}^T(\Omega_S^J)$. For each $w \in S$, $\mathcal{O}_{X_w^J}$ belongs to $\operatorname{Coh}^T(\Omega_S^J)$ and therefore defines a class $[\mathcal{O}_{X_w^J}] \in K^T(\Omega_S^J)$. Define

$$K^T(X^J) = \varprojlim_S K^T(\Omega_S^J).$$

One may show (as in [KS09] for the case where $J = \emptyset$) that

$$K^T(X^J) \cong \prod_{w \in W^J} K^T(\mathrm{pt})[\mathcal{O}_{X_w^J}]. \tag{3.2}$$

Recall that $K^T(\mathrm{pt}) \cong R(T) \cong \mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z}e^\lambda$. Elements of $K^T(X^J)$ are possibly infinite $K^T(\mathrm{pt})$-linear combinations of equivariant Schubert classes $[\mathcal{O}_{X_w^J}]$.

3.3 Restriction to T-fixed points

For $w \in W^J$, let $i_w^J : \{\mathrm{pt}\} \to W^J \cong (X^J)^T \subset X^J$ be the inclusion with image $\{wP_J^-/P_J^-\}$. Restriction to the T-fixed points induces an injective $R(T)$-algebra homomorphism

$$\begin{array}{ccc} K^T(X^J) & \xrightarrow{\mathrm{res}^J} & K^T((X^J)^T) \cong K^T(W^J) \cong \operatorname{Fun}(W^J, R(T)), \\ c & \longmapsto & (w \longmapsto i_w^{J*}(c)) \end{array} \tag{3.3}$$

(see [HHH05, KK90]), where $\operatorname{Fun}(W^J, R(T))$ is the $R(T)$-algebra of functions $W^J \to R(T)$ with pointwise multiplication and $R(T)$-action

$$(q\psi)(w) = q\psi(w)$$

for $q \in R(T)$, $\psi \in \operatorname{Fun}(W^J, R(T))$ and $w \in W^J$. Let $\iota_J : \operatorname{Fun}(W^J, R(T)) \to \operatorname{Fun}(W, R(T))$ be defined by extending functions to be constant on cosets in W/W_J:

$$\iota_J(\psi)(w) = \psi(w')$$

for $\psi \in \operatorname{Fun}(W^J, R(T))$, where $w' \in W^J$ is such that $w'W_J = wW_J$.

Define $\Psi^J \subset \operatorname{Fun}(W, R(T))$ by $\psi \in \Psi^J$ if and only if ψ is in the image of ι_J and

$$\psi(r_\alpha w) - \psi(w) \in (1 - e^\alpha)R(T) \quad \text{for all } w, r_\alpha w \in W,\ \alpha \in \Phi^{\mathrm{re}}. \tag{3.4}$$

We call this the GKM condition[1] for $K^T(X^J)$.

[1] The corresponding criterion was proved in [GKM98] for equivariant cohomology for more general spaces, commonly called GKM spaces. For Kac–Moody flag ind-schemes the criterion follows directly from results in [KK90]. See also [HHH05] for more general cohomology theories and spaces.

Theorem 3.1 [HHH05, KK90].

$$K^T(X^J) \cong \Psi^J.$$

For the sake of completeness, we include a proof of Theorem 3.1. For $v \in W^J$, define $\psi^v_J \in \mathrm{Fun}(W^J, R(T))$ to be the image of $[\mathcal{O}_{X^J_v}]$:

$$\psi^v_J(w) = i^{J*}_w([\mathcal{O}_{X^J_v}]) \quad \text{for } v, w \in W^J. \tag{3.5}$$

When $J = \emptyset$ we shall write $X = X^\emptyset$, $\Psi = \Psi^\emptyset$ and so on, suppressing $\emptyset$ in the notation. Observe that the definition of $\Psi = \Psi^\emptyset$ in this section agrees with the definition (2.9) by Proposition 2.6. Provisionally, for $J = \emptyset$, we write $\psi^v_\emptyset$ for the functions defined by (3.5) and show that they agree with the functions defined by (2.10) using the K-NilHecke ring.

Theorem 3.2. *For all $v \in W$, we have $\psi^v = \psi^v_\emptyset$.*

3.4 Push–pull and y_i

Fix $i \in I$. For the singleton $J = \{i\}$, let $P^-_i = P^-_J$ and $X^i = X^J$, and let $p_i : X \to X^i$ be the projection, which is a $\mathbb{P}^1$-bundle. In [KS09] it is shown that

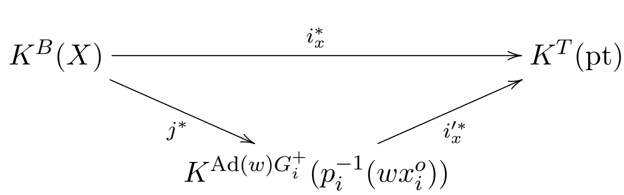
$$p_i^* p_{i\,*}([\mathcal{O}_{X_v}]) = \begin{cases} [\mathcal{O}_{X_{vr_i}}] & \text{if } vr_i < v, \\ [\mathcal{O}_{X_v}] & \text{if } vr_i > v. \end{cases} \tag{3.6}$$

Proposition 3.3. *The map $\psi \mapsto y_i \cdot \psi$ is an $R(T)$-module endomorphism of $\mathrm{Fun}(W, R(T))$ such that the following diagram commutes.*

$$\begin{CD} K^T(X) @>{\mathrm{res}}>> \mathrm{Fun}(W, R(T)) \\ @V{p_i^* p_{i\,*}}VV @VV{y_i \cdot -}V \\ K^T(X) @>>{\mathrm{res}}> \mathrm{Fun}(W, R(T)) \end{CD}$$

Proof. Let $x^o_i \in X_i$ be the point P^-_i/P^-_i. Let $G_i \supset G^+_i \supset T$ be the subgroups with $\mathrm{Lie}(G_i) = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}$ and $\mathrm{Lie}(G^+_i) = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_i}$.

Now let $w \in W$ with $wr_i < w$. Then $\mathrm{Ad}(w)G^+_i \subset B$, and it stabilizes wx^o_i and $p_i^{-1}(wx^o_i)$. Let $j : p_i^{-1}(wx^o_i) \to X$ be the inclusion. Then for $\mathcal{F} \in \mathrm{Coh}^B(X)$, the first left derived functor $L_1 j^* \mathcal{F}$ is $\mathrm{Ad}(w)G^+_i$-equivariant. For $x \in \{w, wr_i\}$, let i'_x be the inclusion $\{\mathrm{pt}\} \to xx^o_i \subset p_i^{-1}(wx^o_i)$. Hence we have the commutative diagram



and the following isomorphisms that forget down to the Levi:

$$K^B(X) \cong K^T(X),$$
$$K^{\mathrm{Ad}(w)G^+_i}(p_i^{-1}(wx^o_i)) \cong K^T(p_i^{-1}(wx^o_i)).$$

This allows reduction to the case of $p_i^{-1}(wx^o_i) \cong \mathbb{P}^1$, where the result is standard; see [CG97, Corollary 6.1.17]. □

Proof of Theorem 3.2. We show that the functions $\psi_{\emptyset}^{v}$ satisfy the recurrence in Remark 2.2.

Since $X_{\mathrm{id}} = X$, it follows that $\psi_{\emptyset}^{\mathrm{id}}(\mathrm{id}) = 1$.

Let $v \in W$. By (3.1), we have $w \in X_v$ if and only if $w \geqslant v$. Therefore

$$\psi_{\emptyset}^{v}(w) = 0 \quad \text{unless } v \leqslant w.$$

In particular, $\psi_{\emptyset}^{v}(\mathrm{id}) = 0$ if $v \neq \mathrm{id}$. Therefore the values $\psi_{\emptyset}^{v}(w)$ satisfy the base case of the recurrence. Proving (2.12) holds for $\psi_{\emptyset}^{v}$ is equivalent to showing that

$$y_i \cdot \psi_{\emptyset}^{v} = \begin{cases} \psi_{\emptyset}^{vr_i} & \text{if } vr_i < v, \\ \psi_{\emptyset}^{v} & \text{if } vr_i > v. \end{cases}$$

But this holds by Proposition 3.3 and (3.6). □

Let $p_J : X \to X^J$ be the projection. Then there is the following commutative diagram of injective $R(T)$-algebra maps, where the horizontal maps are restriction maps as in (3.3).

$$\begin{array}{ccc} K^T(X^J) & \xrightarrow{\mathrm{res}^J} & \mathrm{Fun}(W^J, R(T)) \\ {\scriptstyle p_J^*}\downarrow & & \downarrow{\scriptstyle \iota_J} \\ K^T(X) & \xrightarrow[\mathrm{res}]{} & \mathrm{Fun}(W, R(T)) \end{array}$$

We have $p_J^*([\mathcal{O}_{X_v^J}]) = [\mathcal{O}_{X_v}]$ and $\iota_J(\psi_J^v) = \psi^v$ for $v \in W^J$.

Proof of Theorem 3.1. For the $J = \emptyset$ case, the result follows from (3.2), Theorem 3.2 and Proposition 2.6. For $J \neq \emptyset$, let $\psi = \sum_w a_w \psi^w$ be in the image of ι_J. It suffices to show that $a_w = 0$ for $w \notin W^J$. Since T_i is a $Q(T)$-multiple of $r_i - 1$, we have $T_i \cdot \psi = 0$ whenever $i \in J$. Suppose that $a_w \neq 0$ for some $w \notin W^J$. Pick such a w with minimal length, and let $i \in J$ be such that $wr_i < w$. From Lemma 2.2 and $T_i = y_i - 1$, we deduce that the coefficient of ψ^{wr_i} in $T_i \cdot \psi$ is non-zero, which is a contradiction. □

4. The affine flag manifold and affine Grassmannian

We now specialize our constructions to the case of an affine root system, and consider the thick affine flag manifold X_{af} and thick affine Grassmannian Gr_G and their equivariant K-cohomology. However, instead of using the full affine torus $T_{\mathrm{af}} \subset G_{\mathrm{af}}$, we shall use the torus $T \subset G$ and consider $K^T(\mathrm{Gr}_G)$. We give a *small-torus GKM condition*, which is the K-theoretic analogue of a result of Goresky *et al.* [GKM04] in cohomology.

4.1 The affine flag manifold

We fix notation specific to affine root systems and their associated finite root systems.

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ such that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$, $\mathfrak{b}$ is a Borel subalgebra, and $\mathfrak{t}$ is a Cartan subalgebra. Also, take a Dynkin node set I, a finite Weyl group W, simple reflections $\{r_i \mid i \in I\}$, a weight lattice $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i \subset \mathfrak{t}^*$ with fundamental weights ω_i, a root lattice $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{t}^*$ with simple roots α_i, and a coroot lattice $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{t}$.

Let $G \supset B \supset T$ such that G is a simple and simply connected algebraic group over $\mathbb{C}$ with $\mathrm{Lie}(G) = \mathfrak{g}$, B is a Borel subgroup, and T is a maximal algebraic torus.

Let $\mathfrak{g}_{\mathrm{af}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Kac–Moody algebra with canonical simple subalgebra $\mathfrak{g}$, canonical central element c and degree derivation d. Let $\mathfrak{g}_{\mathrm{af}} = \mathfrak{n}_{\mathrm{af}}^+ \oplus \mathfrak{t}_{\mathrm{af}} \oplus \mathfrak{n}_{\mathrm{af}}^-$

be the triangular decomposition with affine Cartan subalgebra $\mathfrak{t}_{\mathrm{af}}$. Let $I_{\mathrm{af}} = \{0\} \cup I$ be the affine Dynkin node set. Let $\{a_i \in \mathbb{Z}_{>0} \mid i \in I_{\mathrm{af}}\}$ be the unique collection of relatively prime positive integers giving a dependency for the columns of the affine Cartan matrix $(a_{ij})_{i,j \in I_{\mathrm{af}}}$. Then $\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i$ is the null root. The affine weight lattice is given by $P_{\mathrm{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i \in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i \subset \mathfrak{t}^*_{\mathrm{af}}$ where $\{\Lambda_i \mid i \in I_{\mathrm{af}}\}$ are the affine fundamental weights. Let Q_{af} and $Q^\vee_{\mathrm{af}}$ be the affine root and coroot lattices. Let W_{af} be the affine Weyl group, with simple reflections r_i for $i \in I_{\mathrm{af}}$. Considering the subset I of I_{af}, $W = W_I \subset W_{\mathrm{af}}$ and W^I_{af} is the set of minimal-length coset representatives in W_{af}/W. We have $W_{\mathrm{af}} \cong W \ltimes Q^\vee$ with $\lambda \in Q^\vee$ written as $t_\lambda \in W_{\mathrm{af}}$. There is a bijection $W^I_{\mathrm{af}} \to Q^\vee$ sending $w \in W^I_{\mathrm{af}}$ to $\lambda \in Q^\vee$, where λ is defined by $wW = t_\lambda W$. Let $\lambda^- \in W \cdot \lambda$ be antidominant and $u \in W$ the shortest element such that $u\lambda^- = \lambda$. Then $w = t_\lambda u$ and $\ell(w) = \ell(t_\lambda) - \ell(u)$.

Let $\Phi_{\mathrm{af}} = \mathbb{Z}\delta \cup (\mathbb{Z}\delta + \Phi)$ be the set of affine roots. The affine real roots are given by $\Phi^{\mathrm{re}}_{\mathrm{af}} = \mathbb{Z}\delta + \Phi$. Let $\Phi^{\pm}_{\mathrm{af}}$ be the sets of positive and negative affine roots. The set of positive affine real roots is defined by $\Phi^{+\mathrm{re}}_{\mathrm{af}} = \Phi^+_{\mathrm{af}} \cap \Phi^{\mathrm{re}}_{\mathrm{af}} = \Phi^+ \cup (\mathbb{Z}_{>0}\delta + \Phi)$. A typical real root $\alpha + m\delta \in \Phi^{\mathrm{re}}_{\mathrm{af}}$, with $\alpha \in \Phi$ and $m \in \mathbb{Z}$, has associated reflection $r_{\alpha+m\delta} = r_\alpha t_{m\alpha^\vee} \in W_{\mathrm{af}}$.

Let $G_{\mathrm{af}} \supset P^-_I \supset B^-_{\mathrm{af}} \supset T_{\mathrm{af}}$ be the schemes of § 3.1 associated with $\mathfrak{g}_{\mathrm{af}}$, where P^-_I is the maximal parabolic group scheme for the subset of Dynkin nodes $I \subset I_{\mathrm{af}}$ and B^-_{af} is the negative affine Borel group; then $X_{\mathrm{af}} = G_{\mathrm{af}}/B^-_{\mathrm{af}}$ is the thick affine flag manifold and $\mathrm{Gr}_G = X^I_{\mathrm{af}} = G_{\mathrm{af}}/P^-_I$ the thick affine Grassmannian.

4.2 Equivariant K-theory for affine flags with small-torus action

Following Peterson [Pet97] and Goresky *et al.* [GKM04] in the cohomology case, we consider the action of the smaller torus $T = T_{\mathrm{af}} \cap G$. The goal is to formulate and prove the analogue of Theorem 3.1 for $K^T(X_{\mathrm{af}})$ and $K^T(\mathrm{Gr}_G)$. We let $\Psi'_{\mathrm{af}} \subset \mathrm{Fun}(W_{\mathrm{af}}, R(T_{\mathrm{af}}))$ denote the ring defined by (2.9) for the affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$, so that $\Psi'_{\mathrm{af}} \cong K^{T_{\mathrm{af}}}(X_{\mathrm{af}})$.

The natural projection $P_{\mathrm{af}} \to P$ of weight lattices is surjective with kernel $\mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$. It induces the projections

$$\mathbb{Z}[P_{\mathrm{af}}] \xrightarrow{\phi} \mathbb{Z}[P] \xrightarrow{\phi_0} \mathbb{Z}$$

and a commutative diagram

$$\begin{array}{ccc} K^{T_{\mathrm{af}}}(X_{\mathrm{af}}) & \xrightarrow{\mathrm{res}'} & \Psi'_{\mathrm{af}} \\ \downarrow{\scriptstyle \mathrm{For}} & & \downarrow{\scriptstyle \phi\circ-} \\ K^T(X_{\mathrm{af}}) & \xrightarrow[\mathrm{res}]{} & \mathrm{Fun}(W_{\mathrm{af}}, R(T)) \end{array} \tag{4.1}$$

where the horizontal maps are restrictions to $W_{\mathrm{af}} = X^{T_{\mathrm{af}}}_{\mathrm{af}} \subset X^T_{\mathrm{af}}$ and the vertical map For regards a T_{af}-equivariant $O_{X_{\mathrm{af}}}$-module as a T-equivariant one. We change notation slightly, writing the Schubert classes as $\psi'^v \in \Psi'_{\mathrm{af}}$ and defining $\psi^v := \phi \circ \psi'^v \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$.

The following definition is inspired by the analogous cohomological condition in [GKM04, Theorem 9.2]. A function $\psi \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ can be extended by linearity to give a function $\psi' \in \mathrm{Fun}(\bigoplus_{w \in W_{\mathrm{af}}} R(T) \cdot w, R(T))$. In the following definition we abuse notation by identifying ψ with ψ'.

Definition 4.1. We say that $\psi \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ satisfies the *small-torus Grassmannian GKM condition* if

$$\psi((1 - t_{\alpha^\vee})^d w) \in (1 - e^\alpha)^d R(T) \quad \text{for all } d \in \mathbb{Z}_{>0},\ w \in W_{\mathrm{af}} \text{ and } \alpha \in \Phi. \tag{4.2}$$

We say that $\psi \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ satisfies the *small-torus GKM condition* if, in addition to (4.2), we have

$$\psi((1-t_{\alpha^\vee})^{d-1}(1-r_\alpha)w) \in (1-e^\alpha)^d R(T) \quad \text{for all } d \in \mathbb{Z}_{>0}, w \in W_{\mathrm{af}} \text{ and } \alpha \in \Phi. \tag{4.3}$$

Let Ψ_{af} be the set of $\psi \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ that satisfy the small-torus GKM condition, and let Ψ_{af}^I be the set of $\psi \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ that are constant on cosets wW for $w \in W_{\mathrm{af}}$ and satisfy the small-torus Grassmannian GKM condition.

LEMMA 4.2. *Suppose ψ satisfies (4.2) and let $J := (1-e^\alpha)^d R(T)$. Then*

$$\psi((1-t_{\alpha^\vee})^{d-1}w) \quad \text{and} \quad \psi((1-t_{\alpha^\vee})^{d-1}t_{p\alpha^\vee}w)$$

are congruent modulo the ideal J for all $p \in \mathbb{Z}$.

Proof. By (4.2),

$$J \ni \psi((1-t_{\alpha^\vee})^d w) = \psi((1-t_{\alpha^\vee})^{d-1}w) - \psi((1-t_{\alpha^\vee})^{d-1}t_{\alpha^\vee}w)$$

so that the assertion holds for $p = 1$. Repeating the same argument for $\psi((1-t_{\alpha^\vee})^{d-1}t_{\alpha^\vee}w)$ yields the lemma for all $p \in \mathbb{Z}_{\geqslant 0}$. Replacing w by $t_{-p\alpha^\vee}w$ gives the statement for all $p \in \mathbb{Z}$. □

THEOREM 4.3.

(i) $K^T(X_{\mathrm{af}}) \cong \Psi_{\mathrm{af}} = \prod_{v \in W_{\mathrm{af}}} R(T)\,\psi^v$.

(ii) $K^T(\mathrm{Gr}_G) \cong \Psi_{\mathrm{af}}^I = \prod_{v \in W_{\mathrm{af}}^I} R(T)\,\psi^v$.

Proof. Owing to Lemmata 2.3 and 2.5, the set $\{\psi^v \mid v \in W_{\mathrm{af}}\}$ is independent over $R(T)$. Arguing as in [KS09], one can show that $K^T(X_{\mathrm{af}})$ consists of possibly infinite $R(T)$-linear combinations of the $[\mathcal{O}_{X_v}]$. By the commutativity of the diagram (4.1), we conclude that

$$K^T(X_{\mathrm{af}}) \cong \prod_{v \in W_{\mathrm{af}}} R(T)[\mathcal{O}_{X_v}],$$

that the map For is surjective, and that res$'$ is injective with image $\prod_{v \in W_{\mathrm{af}}} R(T)\psi^v$. For (i) it remains to show that $\Psi_{\mathrm{af}} = \prod_{v \in W_{\mathrm{af}}} R(T)\psi^v$. Let $v \in W_{\mathrm{af}}$. We first show that $\psi^v \in \Psi_{\mathrm{af}}$. Let $w \in W_{\mathrm{af}}$, $\alpha \in \Phi$ and $d \in \mathbb{Z}_{>0}$. Let $W' \subset W_{\mathrm{af}}$ be the subgroup generated by $t_{\alpha^\vee}$ and r_α; it is isomorphic to the affine Weyl group of SL_2. Define the function $f : W' \to R(T)$ by $f(x) = \psi'^v(xw)$. Since ψ'^v satisfies the big-torus GKM condition (3.4) for X_{af}, f satisfies (3.4) for a copy of the SL_2 affine flag variety X'. Therefore f is a possibly infinite $R(T)$-linear combination of Schubert classes in X'. By Propositions 4.4 and 4.5, proved below, $\phi \circ f$ satisfies the small-torus GKM condition for X'. It follows that $\psi^v \in \Psi_{\mathrm{af}}$.

Conversely, suppose that $\psi \in \Psi_{\mathrm{af}}$. We show that $\psi \in \prod_{v \in W_{\mathrm{af}}} R(T)\psi^v$. Let $x = t_\lambda u \in \mathrm{Supp}(\psi)$ be of minimal length, with $u \in W$ and $\lambda \in Q^\vee$. It suffices to show that

$$\psi(x) \in \psi^x(x)R(T)$$

because, upon defining $\psi' \in \Psi_{\mathrm{af}}$ by $\psi' = \psi - (\psi(x)/\psi^x(x))\psi^x$, we have $\mathrm{Supp}(\psi') \subsetneq \mathrm{Supp}(\psi)$, and by repeating this we may write ψ as a $R(T)$-linear combination of the ψ^x.

The elements $\{1 - e^\alpha \mid \alpha \in \Phi^+\}$ are relatively prime in $R(T)$. Letting $\alpha \in \Phi^+$, by Lemma 2.5 it suffices to show that

$$\psi(x) \in J := (1-e^\alpha)^d R(T),$$

where $d = |\mathrm{Inv}_\alpha(x)|$ with $\mathrm{Inv}_\alpha(x)$ being the set of roots in $\mathrm{Inv}(x)$ of the form $\pm\alpha + k\delta$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Note that for $\beta \in \Phi_{\mathrm{af}}^{+\mathrm{re}}$, $\beta \in \mathrm{Inv}(x)$ if and only if $x^{-1} \cdot \beta \in -\Phi_{\mathrm{af}}^{+\mathrm{re}}$. We have

$$x^{-1} \cdot (\pm\alpha + k\delta) = u^{-1} t_{-\lambda} \cdot (\pm\alpha + k\delta) = \pm u^{-1}\alpha + (k \pm \langle \lambda, \alpha \rangle)\delta.$$

Hence

$$\mathrm{Inv}_\alpha(x) = \begin{cases} \{\alpha, \alpha+\delta, \dots, \alpha - (\langle\lambda,\alpha\rangle + \chi(\alpha \notin \mathrm{Inv}(u)))\delta\} & \text{if } \langle\lambda,\alpha\rangle \leqslant 0, \\ \{-\alpha+\delta, -\alpha+2\delta, \dots, -\alpha + (\langle\lambda,\alpha\rangle - \chi(\alpha \in \mathrm{Inv}(u)))\delta\} & \text{if } \langle\lambda,\alpha\rangle > 0. \end{cases}$$

Suppose first that $\langle\lambda,\alpha\rangle > 0$. Then $d = \langle\lambda,\alpha\rangle - \chi(\alpha \in \mathrm{Inv}(u))$. Applying (4.3) to $y = t_{(1-d)\alpha^\vee} x$, we get $Z_1 \in J$ where

$$\begin{aligned} Z_1 &= \psi((1-t_{\alpha^\vee})^{d-1}(1-r_\alpha)y) \\ &= (-1)^{d-1}\psi((1-t_{-\alpha^\vee})^{d-1}x) - \psi((1-t_{\alpha^\vee})^{d-1} r_\alpha y) \\ &= (-1)^{d-1}\psi(x) - \psi((1-t_{\alpha^\vee})^{d-1} r_\alpha y). \end{aligned}$$

The last equality holds by the assumption on $\mathrm{Supp}(\psi)$ and a calculation of $\mathrm{Inv}_\alpha(r_{\alpha+d\delta}x)$, giving $x > r_{\alpha+d\delta}x > t_{-k\alpha^\vee}x$ for all $k \in [1, d-1]$. By Lemma 4.2, we have $Z_2 \in J$ where

$$Z_2 = \psi((1-t_{\alpha^\vee})^{d-1} r_\alpha y) - \psi((1-t_{\alpha^\vee})^{d-1} t_{p\alpha^\vee} r_\alpha y)$$

for any $p \in \mathbb{Z}$. Thus

$$Z_1 + Z_2 = (-1)^{d-1}\psi(x) - \psi((1-t_{\alpha^\vee})^{d-1} t_{p\alpha^\vee} r_\alpha y) \in J.$$

By the assumption on $\mathrm{Supp}(x)$ and the calculation of $\mathrm{Inv}_\alpha(x)$,

$$\psi((1-t_{\alpha^\vee})^{d-1} t_{p\alpha^\vee} r_\alpha y) = 0$$

for $p = 2-d$. It follows that $\psi(x) \in J$.

Now suppose $\langle\lambda,\alpha\rangle \leqslant 0$. By the previous case, we may assume that $t_{d\alpha^\vee}x \notin \mathrm{Supp}(\psi)$. Thus

$$\psi(x) = \psi((1-t_{\alpha^\vee})^d x) \in J$$

by induction on $\mathrm{Supp}(\psi)$ and (4.2). This proves (i).

For (ii), it suffices to show that $\psi \in \Psi_{\mathrm{af}}^I$ if and only if $\psi \in \prod_{v \in W_{\mathrm{af}}^I} R(T)\psi^v$. First, let $\psi \in \Psi_{\mathrm{af}}^I$. Let $w = t_\lambda u \in W_{\mathrm{af}}$ with $\lambda \in Q^\vee$ and $u \in W$, and take $\alpha \in \Phi$. We shall verify that ψ satisfies the small-torus GKM condition. We have

$$r_\alpha w = r_\alpha t_\lambda u = t_{r_\alpha(\lambda)} r_\alpha u = t_{-\langle\lambda,\alpha\rangle\alpha^\vee} t_\lambda r_\alpha u.$$

Since by assumption ψ is constant on cosets W_{af}/W, we have

$$\psi((1-t_{\alpha^\vee})^{d-1}(1-r_\alpha)w) = \psi((1-t_{\alpha^\vee})^{d-1}(1-t_{-\langle\lambda,\alpha\rangle\alpha^\vee})t_\lambda).$$

But $1 - t_{k\alpha^\vee}$ is divisible by $1 - t_{\alpha^\vee}$ for any $k \in \mathbb{Z}$. Therefore ψ satisfies the small-torus GKM condition because it satisfies the Grassmannian one. Part (i) and the fact that ψ is constant on cosets W_{af}/W implies that $\psi \in \prod_{v \in W_{\mathrm{af}}^I} R(T)\psi^v$.

Conversely, it suffices to show that for every $v \in W_{\mathrm{af}}^I$ we have $\psi^v \in \Psi_{\mathrm{af}}^I$. But this follows from part (i) and the fact that for such v, ψ^v is constant on cosets W_{af}/W. □

4.3 The small-torus GKM condition for $\widehat{\mathfrak{sl}}_2$

In this section we prove that the Schubert classes ψ^v for $\widehat{\mathfrak{sl}}_2$ satisfy the small-torus GKM condition of Definition 4.1. To this end, we first derive explicit expressions for the ψ^v.

For $i \in \mathbb{Z}_{\geqslant 0}$, let

$$\begin{aligned} \sigma_{2i} = (r_1 r_0)^i, \quad \sigma_{-2i} = (r_0 r_1)^i, \\ \sigma_{2i+1} = r_0 \sigma_{2i}, \quad \sigma_{-(2i+1)} = r_1 \sigma_{-2i}. \end{aligned} \tag{4.4}$$

Then $\ell(\sigma_j) = |j|$ for $j \in \mathbb{Z}$, $W_{\mathrm{af}}^I = \{\sigma_j \mid j \in \mathbb{Z}_{\geqslant 0}\}$, and

$$\sigma_{2i} = t_{-i\alpha^\vee} \quad \text{for } i \in \mathbb{Z}.$$

Let $\psi_j^i := \psi^{\sigma_i}(\sigma_j)$ for $i, j \in \mathbb{Z}$, where we have set $\delta = 0$. We set $x = e^\alpha$ and let $S_{\leqslant a}$ be the sum $h_0 + h_1 + \cdots + h_a$ of homogeneous symmetric functions. We write $S^i_{\leqslant a}(x)$ to mean $S_{\leqslant a}[x, x, \ldots, x]$ where there are i copies of x. For $i, a \in \mathbb{Z}$ such that $a, m \geqslant 0$, we have

$$\psi^m_{2i+2a} = (1-x)^m S^m_{\leqslant a}(x) = \psi^m_{-2i-2a-1} \quad \text{for } m = 2i \text{ or } 2i-1, \tag{4.5}$$

$$\psi^m_{2i+2a+1} = (1-x^{-1})^m S^m_{\leqslant a}(x^{-1}) = \psi^m_{-2i-2a-2} \quad \text{for } m = 2i \text{ or } 2i+1, \tag{4.6}$$

and zero otherwise. Furthermore,

$$\psi^{-m}_{-i}(x) = \psi^m_i(x^{-1}).$$

These relations are easily proved by induction using the left- and right-hand recurrences for the localization of Schubert classes together with the recurrence

$$S^i_{\leqslant a}(x) = x S^i_{\leqslant a-1}(x) + S^{i-1}_{\leqslant a}(x).$$

We also have the explicit formula

$$S^i_{\leqslant a}(x) = \sum_{j=0}^a x^j \binom{j+i-1}{i-1}. \tag{4.7}$$

PROPOSITION 4.4. *For all $d \geqslant 1$, $m \in \mathbb{Z}$ and $w \in W_{\mathrm{af}}$ we have*

$$\psi^m((1-t_{\alpha^\vee})^d w) \in (1-x)^d \mathbb{Z}[x^\pm].$$

Proof. We prove the claim for $m = 2i$ and for the ranges $t_{(-i-a)\alpha^\vee}$ to $t_{(i+1+b)\alpha^\vee}$ for $a, b \in \mathbb{Z}_{\geqslant 0}$. The other cases are similar. Let $d = (i+a) + (i+1+b) = 2i + a + b + 1$. We must show that

$$Z := \sum_{k=0}^d (-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k} \in (1-x)^d \mathbb{Z}[x^\pm].$$

Since $\psi^{2i}_{2p} = 0$ for $-2i-2 < 2p < 2i$,

$$\begin{aligned} Z &= \left(\sum_{k=0}^b + \sum_{k=2i+1+b}^d\right)(-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k} \\ &= \sum_{k=0}^b (-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k} + \sum_{k=0}^a (-1)^{k+2i+1+b} \binom{d}{2i+1+b+k} \psi^{2i}_{2i+2k} \\ &= (-1)^b \sum_{k=0}^b (-1)^k \binom{d}{b-k} \psi^{2i}_{-2i-2-2k} - (-1)^b \sum_{k=0}^a (-1)^k \binom{d}{a-k} \psi^{2i}_{2i+2k}. \end{aligned}$$

Substituting (4.5), (4.6) and (4.7) gives

$$(-1)^b Z = \sum_{k=0}^{b} (-1)^k \binom{d}{b-k} (1-x^{-1})^{2i} \sum_{j=0}^{k} x^{-j} \binom{j+2i-1}{2i-1} \\ - \sum_{k=0}^{a} (-1)^k \binom{d}{a-k} (1-x)^{2i} \sum_{j=0}^{k} x^{j} \binom{j+2i-1}{2i-1}.$$

Therefore we must show that $Z' := (-1)^b Z(1-x)^{-2i}$ is divisible by $(1-x)^{a+b+1}$. Regarding Z' as a function of x, we need to show that its rth derivative at $x=1$ vanishes, for $0 \leqslant r \leqslant a+b$. This yields the identities

$$(-1)^r \sum_{k=0}^{b} (-1)^k \binom{d}{b-k} \sum_{j=0}^{k} \frac{(j+2i+r-1)!}{j!} = \sum_{k=0}^{a} (-1)^k \binom{d}{a-k} \sum_{j=r}^{k} \frac{(j+2i-1)!}{(j-r)!}.$$

Upon shifting the sums on the right-hand side using $j' = j-r$ and $k' = k-r$ and dividing both sides by $(-1)^r(2i+r-1)!$, the inner sums simplify and we obtain

$$\sum_{k=0}^{b} (-1)^k \binom{d}{b-k} \binom{2i+r+k}{k} = \sum_{k=0}^{a-r} (-1)^k \binom{d}{a-r-k} \binom{2i+r+k}{k}.$$

Setting $a' = a-r$, we claim that this sum is equal to $\binom{a'+b}{b} = \binom{a'+b}{a'}$, which is symmetric in a' and b and hence implies equality of the two sides. This can be seen as follows. The coefficient of x^b in $(1+x)^{a'+b}$ is $\binom{a'+b}{a'}$. Alternatively, we can calculate

$$[x^b](1+x)^{a'}(1+x)^b(1+x)^c(1+x)^{-c},$$

where $c = 2i+r+1$ and $(1+x)^{-c}$ is meant to be expanded as a power series in x. Then

$$[x^b](1+x)^{a'}(1+x)^b(1+x)^c(1+x)^{-c} = \sum_{k=0}^{b} [x^{b-k}](1+x)^{a'+b+c}[x^k](1+x)^{-c},$$

which is exactly the sum we wanted to evaluate. □

Proposition 4.5. *For all $d \geqslant 1$, $m \in \mathbb{Z}$ and $w \in W_{\mathrm{af}}$ we have*

$$\psi^m((1-t_{\alpha^\vee})^{d-1}(1-r_\alpha)w) \in (1-x)^d \mathbb{Z}[x^{\pm}].$$

Proof. Note that

$$\psi^m((1-t_{\alpha^\vee})^{d-1}(1-r_\alpha)w) = \psi^m((1-t_{\alpha^\vee})^{d-1}w) - \psi^m((1-t_{\alpha^\vee})^{d-1}r_\alpha w). \tag{4.8}$$

Furthermore, by Proposition 4.4, ψ^m satisfies the small-torus Grassmannian GKM condition. Hence, applying Lemma 4.2 to $\psi^m((1-t_{\alpha^\vee})^{d-1}r_\alpha w)$, we can shift the argument $r_\alpha w$ so that the equalities (4.5) and (4.6) can be used. This implies that (4.8) is zero modulo the ideal $(1-x)^d\mathbb{Z}[x^{\pm}]$. □

4.4 The wrong-way map

There is a natural inclusion map $\iota_I : \Psi^I_{\mathrm{af}} \to \Psi_{\mathrm{af}}$. In the case at hand, there is a map $\varpi : \Psi_{\mathrm{af}} \to \Psi^I_{\mathrm{af}}$, of which ι_I is a section. This map is specific to the case of the affine Grassmannian; it also does not exist if one uses the larger torus T_{af}.

Lemma 4.6. *There is an $R(T)$-module homomorphism $\varpi : \Psi_{\mathrm{af}} \to \Psi_{\mathrm{af}}^I$, defined by $\varpi(\psi)(w) = \psi(t_\lambda)$ for $w \in W_{\mathrm{af}}$, where $\lambda \in Q^\vee$ is such that $wW = t_\lambda W$.*

Proof. Let $\psi \in \Psi_{\mathrm{af}}$, $w = t_\lambda u \in W_{\mathrm{af}}$ with $\lambda \in Q^\vee$, and $u \in W$, $\alpha \in \Phi$ and $d \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned}\varpi(\psi)((1-t_{\alpha^\vee})^d w) &= \varpi(\psi)((1-t_{\alpha^\vee})^d t_\lambda u)\\ &= \psi((1-t_{\alpha^\vee})^d t_\lambda) \in (1-e^\alpha)^d R(T).\end{aligned}$$

□

5. *K*-homology of the affine Grassmannian and the *K*-Peterson subalgebra

Let $\mathbb{K}'$ be the K-NilHecke ring for the affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$ defined via the general construction in § 2. In this section we use the *affine K-NilHecke ring* $\mathbb{K}$, which differs from $\mathbb{K}'$ in the use of $R(T)$ instead of $R(T_{\mathrm{af}})$. Our main result, generalizing work of Peterson [Pet97], gives a Hopf-isomorphism of $K_T(\mathrm{Gr}_G)$ with a commutative subalgebra $\mathbb{L} \subset \mathbb{K}$.

5.1 *K*-homology of the affine Grassmannian

We define the equivariant K-homology $K_T(\mathrm{Gr}_G)$ of the affine Grassmannian to be the continuous dual $K_T(\mathrm{Gr}_G) = \mathrm{Hom}_{R(T)}(K^T(\mathrm{Gr}_G), R(T))$, so that $K_T(\mathrm{Gr}_G)$ is a free $R(T)$-module with basis comprising the Schubert classes ξ_w dual to $[\mathcal{O}_{X_w^I}] \in K^T(\mathrm{Gr}_G)$.

The K-homology $K_T(\mathrm{Gr}_G)$ and K-cohomology $K^T(\mathrm{Gr}_G)$ are equipped with dual Hopf structures, which we now explain, focusing on $K^T(\mathrm{Gr}_G)$ first. Let $K \subset G$ be the maximal compact form, LK the space of continuous loops $S^1 \to K$, and ΩK the space of based loops $(S^1, 1) \to (K, 1)$. Let $T_{\mathbb{R}} = T \cap K$. We denote by $K^{T_{\mathbb{R}}}(\Omega K)$ the equivariant topological K-theory of ΩK. By an (unpublished) well-known result of Quillen (see [HHH05, PS86]), the space ΩK is (equivariantly) weak-homotopy-equivalent to the ind-scheme affine Grassmannian $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. Thus we have $K^{T_{\mathbb{R}}}(\Omega K) = K^{T_{\mathbb{R}}}(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]))$, where $K^{T_{\mathbb{R}}}(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]))$ denotes the topological K-theory of the topological space underlying the ind-scheme $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$.

The topological K-theory $K^{T_{\mathbb{R}}}(\Omega K) \cong K^{T_{\mathbb{R}}}(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]))$ is studied in [KK90], where it is identified with the ring Ψ_{af}^I. More precisely, Kostant and Kumar studied the equivariance with respect to the larger torus T_{af}, but the same argument as in our Theorem 4.3 gives $K^{T_{\mathbb{R}}}(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])) \cong \Psi_{\mathrm{af}}^I$. Thus we obtain the sequence of isomorphisms

$$K^T(\mathrm{Gr}_G) \cong \Psi_{\mathrm{af}}^I \cong K^{T_{\mathbb{R}}}(G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])) \cong K^{T_{\mathbb{R}}}(\Omega K),$$

and all the isomorphisms are compatible with restrictions to fixed points.

The composite map r given by

$$\Omega K \hookrightarrow LK \longrightarrow LK/T_{\mathbb{R}}$$

induces the map

$$K^T(X_{\mathrm{af}}) \cong K^{T_{\mathbb{R}}}(LK/T_{\mathbb{R}}) \xrightarrow{r^*} K^{T_{\mathbb{R}}}(\Omega K) \cong K^T(\mathrm{Gr}_G).$$

One can check, using a fixed-point calculation, that the map ϖ of Lemma 4.6 is related to r^* via the isomorphisms of Theorem 4.3.

The based loop group ΩK has a $T_{\mathbb{R}}$-equivariant multiplication map $\Omega K \times \Omega K \to \Omega K$ given by pointwise multiplication on K, and this induces the structure of a commutative and co-commutative Hopf algebra on $K^{T_{\mathbb{R}}}(\Omega K) \cong K^T(\mathrm{Gr}_G)$. The co-commutativity of $K^{T_{\mathbb{R}}}(\Omega K)$ follows from the fact that it is a homotopy double-loop space (K being already a homotopy loop space).

Via duality, we obtain a dual Hopf-algebra structure on $K_T(\mathrm{Gr}_G)$. For the next result, we label the T_{af}-fixed points of Gr_G by translation elements t_λ.

LEMMA 5.1. *Let $\lambda, \mu \in Q^\vee$, and consider the maps $i_\lambda^*, i_\mu^* : K^T(\mathrm{Gr}_G) \to R(T)$ as elements of $K_T(\mathrm{Gr}_G)$. Then in $K_T(\mathrm{Gr}_G)$ we have*

$$i_\lambda^* \, i_\mu^* = i_{\lambda+\mu}^*.$$

Proof. It suffices to argue in $K^{T_\mathbb{R}}(\Omega K)$. The map $i_\lambda^* \, i_\mu^*$ is induced by the map $\mathrm{pt} \to \Omega K \times \Omega K \to \Omega K$, where the image of the first map is the pair $(t_\lambda, t_\mu) \in \Omega K \times \Omega K$ of fixed points and the second map is multiplication. Treating $t_\lambda, t_\mu : S^1 \to K$ as homomorphisms into K, we see that pointwise multiplication of t_λ and t_μ gives $t_{\lambda+\mu}$. Thus $i_\lambda^* \, i_\mu^* = i_{\lambda+\mu}^*$. □

The antipode of $K_T(\mathrm{Gr}_G)$ is given by $S(i_\lambda^*) = i_{-\lambda}^*$, since the fixed points satisfy $t_\lambda^{-1} = t_{-\lambda}$ in ΩK.

5.2 The affine K-NilHecke ring and K-Peterson subalgebra

Let W_{af} act on the finite-weight lattice P by the (non-faithful) level-zero action $(ut_\lambda \cdot \mu) = u \cdot \mu$ for $u \in W$, $\lambda \in Q^\vee$ and $\mu \in P$.

Let $\mathbb{K}$ be the smash product of the affine 0-Hecke ring $\mathbb{K}_0$ with $R(T)$ (rather than $R(T_{\mathrm{af}})$) using the commutation relations (2.6). We call this the *affine K-NilHecke ring*. The cohomological analogue of $\mathbb{K}$ was studied by Peterson [Pet97]. We have $\mathbb{K} = \bigoplus_{w \in W_{\mathrm{af}}} R(T)\, T_w$.

We now define the map $k : K_T(\mathrm{Gr}_G) \to \mathbb{K}$ by the formula

$$\langle k(\xi), \psi \rangle = \langle \xi, \varpi(\psi) \rangle, \tag{5.1}$$

where $\psi \in \Psi_{\mathrm{af}}$ and ϖ is the wrong-way map of Lemma 4.6. We have used Theorem 4.3(ii) to obtain the pairing on the right-hand side. By letting ψ vary over $\{\psi^v \in \Psi_{\mathrm{af}}\}$, it is clear that (5.1) defines $k(\xi)$ uniquely in $\mathbb{K}$.

We define the *K-Peterson subalgebra* $\mathbb{L} := Z_{\mathbb{K}}(R(T))$ of $\mathbb{K}$ to be the centralizer of $R(T)$ inside $\mathbb{K}$.

LEMMA 5.2. *We have* $\mathrm{Im}(k) = \bigoplus_{\lambda \in Q^\vee} Q(T) t_\lambda \cap \mathbb{K} = \mathbb{L}$.

Proof. For $\lambda \in Q^\vee$, we have $\langle i_\lambda^*, \varpi(\psi) \rangle = \psi(t_\lambda)$, so $k(i_\lambda^*) = t_\lambda \in \mathbb{K}$. Since i_λ^* spans $K_T(\mathrm{Gr}_G)$ (over $Q(T)$), we have thus established the first equality. For the second equality, $\bigoplus_{\lambda \in Q^\vee} Q(T) t_\lambda \cap \mathbb{K} \subseteq Z_{\mathbb{K}}(R(T))$ holds because under the level-zero action t_λ acts on P trivially for all $\lambda \in Q^\vee$. For the other direction, let $a = \sum_{w \in W_{\mathrm{af}}} a_w w \in Z_{\mathbb{K}}(R(T))$ for $a_w \in Q(T)$. Then for all $\mu \in P$ we have

$$0 = e^\mu a - a e^\mu = \sum_{w \in W_{\mathrm{af}}} a_w (e^\mu - e^{w\mu}) w.$$

Therefore, for all $w \in W_{\mathrm{af}}$, either $a_w = 0$ or $w\mu = \mu$ for all $\mu \in P$. Taking μ to be W-regular, we see that the latter holds only for $w = t_\lambda$ with some $\lambda \in Q^\vee$. □

The algebra $\mathbb{L}$ inherits a coproduct $\Delta : \mathbb{L} \to \mathbb{L} \otimes_{R(T)} \mathbb{L}$ from the coproduct of $\mathbb{K}$. (In § 2.6 the coproduct of $\mathbb{K}'$ is given, and it specializes easily to a coproduct for $\mathbb{K}$.) That $\Delta(\mathbb{L}) \subset \mathbb{L} \otimes_{R(T)} \mathbb{L}$ follows from (2.14) and the equality $\mathbb{L} = \bigoplus_{\lambda \in Q^\vee} Q(T) t_\lambda \cap \mathbb{K}$. We make $\mathbb{L}$ a Hopf algebra by defining $S(t_\lambda) = t_{-\lambda}$.

The following results generalize properties of Peterson's j-map in the homology case; see [Lam08, Theorem 4.4].

THEOREM 5.3. *The map $k : K_T(\mathrm{Gr}_G) \to \mathbb{L}$ is a Hopf-isomorphism.*

Proof. To check that a map is a Hopf-morphism, it suffices to check that it is a bialgebra morphism, since the compatibility with antipodes follows as a consequence.

It is clear from the definition that k is injective. Since k is $R(T)$-linear, to check that k is compatible with the Hopf-structure we check the product and coproduct structures on the basis $\{t_\lambda \mid \lambda \in Q^\vee\}$. By Lemma 5.1 we have $k(i^*_{\lambda+\mu}) = k(i^*_\lambda\, i^*_\mu) = t_\lambda\, t_\mu = t_{\lambda+\mu}$, so k is an algebra morphism. That k is a coalgebra morphism follows from an argument similar to that used in proving Lemma 2.8 and Proposition 2.9. Thus $k : K_T(\mathrm{Gr}_G) \to \mathbb{L}$ is a Hopf-isomorphism. □

THEOREM 5.4. *For each $w \in W^I_{\mathrm{af}}$, there is a unique element $k_w \in \mathbb{L}$ of the form*

$$k_w = T_w + \sum_{v \in W_{\mathrm{af}} \setminus W^I_{\mathrm{af}}} k^v_w T_v \tag{5.2}$$

for $k^v_w \in R(T)$. Furthermore, $k_w = k(\xi_w)$ and $\mathbb{L} = \bigoplus_{w \in W^I_{\mathrm{af}}} R(T)\, k_w$.

Proof. Since the Schubert basis $\{\xi_w \mid w \in W^I_{\mathrm{af}}\}$ is a $R(T)$-basis of $K_T(\mathrm{Gr}_G)$, upon setting $k_w = k(\xi_w)$ we obtain, by Theorem 5.3, a $R(T)$-basis of $\mathbb{L}$. By (5.1) and the fact that $\varpi(\psi^v) = \psi^v$ for $v \in W^I_{\mathrm{af}}$, we obtain (5.2). Finally, the element $k_w \in \mathbb{L}$ is unique because the set $\{T_w \mid w \in W^I_{\mathrm{af}}\}$ is linearly independent. □

Define the T-equivariant K-homological Schubert structure constants $d^w_{uv} \in R(T)$ for $K_T(\mathrm{Gr}_G)$ by

$$k_u k_v = \sum_{w \in W^I_{\mathrm{af}}} d^w_{uv} k_w \tag{5.3}$$

where $u, v \in W^I_{\mathrm{af}}$. Since $k_u \in Z_{\mathbb{K}}(R(T))$, we have

$$k_u k_v = k_u \sum_{y \in W_{\mathrm{af}}} k^y_v\, T_y = \sum_{y \in W_{\mathrm{af}}} k^y_v\, k_u\, T_y = \sum_{x,y \in W_{\mathrm{af}}} k^y_v\, k^x_u\, T_x\, T_y.$$

Applying ψ^w with $w \in W^I_{\mathrm{af}}$ and using (5.2) gives

$$d^w_{uv} = \sum_{x,y \in W_{\mathrm{af}}} k^y_v\, k^x_u\, \psi^w(T_x T_y).$$

Since $w \in W^I_{\mathrm{af}}$, $\psi^w(T_x T_y) = 0$ unless $y \in W^I_{\mathrm{af}}$. But, for $y \in W^I_{\mathrm{af}}$, we have $k^y_v = \delta_{yv}$ by (5.2). Therefore

$$d^w_{uv} = \sum_{\substack{x \in W_{\mathrm{af}} \\ T_x T_v = \pm T_w}} (-1)^{\ell(w)-\ell(v)-\ell(x)} k^x_u. \tag{5.4}$$

6. The K-affine Fomin–Stanley algebra and K-homology of the affine Grassmannian

In this section we reduce to the non-equivariant setting. Our main result (Theorem 6.4) describes the specialization at zero of $\mathbb{L}$. We will rely on the corresponding known statements from the cohomological setting, in particular [Lam08, Proposition 5.3].

6.1 K-affine Fomin–Stanley algebra

Define $\phi_0 : R(T) \to \mathbb{Z}$ by setting $\phi_0(e^\lambda) = 1$ and extending by linearity. Define $\phi_0 : \mathbb{K} \to \mathbb{K}_0$ by $\phi_0(a) = \sum_{w\in W} \phi_0(a_w)T_w$, where $a = \sum_{w\in W} a_w T_w$ with $a_w \in R(T)$.

The *K-affine Fomin–Stanley algebra* is defined as

$$\mathbb{L}_0 = \{b \in \mathbb{K}_0 \mid \phi_0(bq) = \phi_0(q)b \text{ for all } q \in R(T)\} \subset \mathbb{K}_0.$$

The cohomological analogue of $\mathbb{L}_0$ was defined in [Lam08].

Lemma 6.1. *Suppose that $a \in \mathbb{L}$. Then $\phi_0(a) \in \mathbb{L}_0$.*

Proof. $\phi_0(ae^\lambda) = \phi_0(e^\lambda a) = \phi_0(a)$. □

In what follows, we shall use the notation (such as A_w) for the cohomological nilHecke ring. We refer the reader to Appendix A for a review of this notation. Let $\lessdot$ denote the covering relation in Bruhat order.

Lemma 6.2. *Let $v \lessdot w$ in W_{af}. Then for each $\lambda \in P$ we have*

$$\phi_0\langle T_w\, e^\lambda, \psi^v\rangle = \phi_0\langle A_w\lambda, \xi^v\rangle = \langle A_w\lambda, \xi^v\rangle.$$

Proof. Write $v = wr_\alpha$. By [Hum90] there exists a length-additive factorization of the form $w = u_1 r_i u_2$ for some $i \in I_{\mathrm{af}}$ such that $v = u_1u_2$ and $\alpha = u_2^{-1}\alpha_i$. We have

$$\phi_0\langle T_w\, e^\lambda, \psi^v\rangle = \phi_0\psi^v(T_w e^\lambda) = \phi_0(u_1 \cdot T_i \cdot e^{u_2\cdot\lambda}) = \phi_0\left(\frac{e^{r_iu_2\lambda} - e^{u_2\lambda}}{1 - e^{\alpha_i}}\right),$$

since $\phi_0(wq) = \phi_0(q)$ for all $w \in W_{\mathrm{af}}$ and $q \in R(T)$. Therefore

$$\phi_0\psi^v(T_we^\lambda) = \langle \alpha_i^\vee, u_2\lambda\rangle = \langle \alpha^\vee, \lambda\rangle = \xi^v(A_w\,\lambda),$$

where we have used (2.1) acting on an exponential for the first equality, W_{af}-equivariance of $\langle\cdot,\cdot\rangle$ for the second equality, and Lemma A.1 for the third. □

Lemma 6.3. *Suppose that $a = \sum_{w\in W_{\mathrm{af}}} a_wT_w \in \mathbb{L}_0$ where $a_w \in \mathbb{Z}$. Let ℓ be maximal so that $a_w \neq 0$ for some w with $\ell(w) = \ell$. Then $a' = \sum_{\ell(w)=\ell} a_wA_w \in \mathbb{B}_0$.*

Proof. We note that for $v \in W_{\mathrm{af}}$ with $\ell(v) = \ell - 1$, we have for each $\lambda \in P$ that

$$\phi_0\;\psi^v(a(e^\lambda - 1)) = \phi_0\;\psi^v\left(\sum_{\ell(w)=\ell} a_wT_w(e^\lambda - 1)\right) = \phi_0\;\xi^v(a'\lambda),$$

using Lemma 6.2. Since $a \in \mathbb{L}_0$, we have $\phi_0(a(e^\lambda - 1)) = 0$ for all λ. Thus $a' \in \mathbb{B}_0$, as claimed. □

Theorem 6.4. *We have $\mathbb{L}_0 = \phi_0(\mathbb{L})$. Furthermore, $\mathbb{L}_0 = \bigoplus_{w\in W_{\mathrm{af}}^I} \mathbb{Z}\,\phi_0(k_w)$ and $\phi_0(k_w)$ is the unique element in $\mathbb{L} \cap (T_w + \bigoplus_{v\in W\setminus W_{\mathrm{af}}^I} \mathbb{Z}\,T_v)$.*

Proof. For $a \in \mathbb{L}$, we have $\phi_0(ae^\lambda) = \phi_0(e^\lambda a) = \phi_0(a)$. Thus $\phi_0(\mathbb{L}) \subset \mathbb{L}_0$. Now suppose that $a = \sum_{w\in W_{\mathrm{af}}} a_wT_w \in \mathbb{L}_0$. Define the support of a to be the $w \in W_{\mathrm{af}}$ such that $a_w \neq 0$. If the support of a contains a Grassmannian element $w \in W_{\mathrm{af}}^I$, then $a - a_w\phi_0(k_w) \in \mathbb{L}_0$, but by Theorem 5.4 its support has fewer Grassmannian elements than does a. So we may suppose that a has no Grassmannian element in its support. By Lemma 6.3, the element a' (as defined in the lemma) lies in $\mathbb{B}_0$ and has no Grassmannian support. By [Lam08, Proposition 5.3], we must have $a' = 0$. Thus $a = 0$. We conclude that $\mathbb{L}_0 = \phi_0(\mathbb{L})$.

Since the $\phi_0(k_w)$, $w \in W_{\mathrm{af}}^I$, are clearly linearly independent, it follows that they form a basis. The last statement follows from Theorem 5.4. □

Some examples of the elements $\phi_0(k_w)$, illustrating Theorem 6.4, are presented in Appendix A.3.3.

Corollary 6.5. *The ring $\mathbb{L}_0$ is commutative.*

Proof. Let $a, b \in \mathbb{L}_0$. By Theorem 6.4, we have $a + a' \in \mathbb{L}$ and $b + b' \in \mathbb{L}$ for some elements a' and b' satisfying $\phi_0(a') = 0 = \phi_0(b')$. Since $\mathbb{L}$ is commutative, we have

$$ab = \phi_0((a + a')(b + b')) = \phi_0((b + b')(a + a')) = ba.$$

□

6.2 Structure constants

We now consider the structure constants in $\mathbb{L}_0$. The next lemma follows from either a direct calculation or Theorem 6.8 below.

Lemma 6.6. *For $a, b \in \mathbb{L}$, we have $\phi_0(ab) = \phi_0(a)\phi_0(b)$.*

Using Lemma 6.6, apply ϕ_0 to (5.3) to get that for $u, v \in W_{\mathrm{af}}^I$,

$$\phi_0(k_u)\phi_0(k_v) = \sum_{w \in W_{\mathrm{af}}^I} \phi_0(d_{uv}^w)\phi_0(k_w). \tag{6.1}$$

In other words, $\phi_0(d_{uv}^w) \in \mathbb{Z}$ are the structure constants for the basis $\{\phi_0(k_v) \mid v \in W_{\mathrm{af}}^I\}$ of $\mathbb{L}_0$.

Conjecture 6.7. For $u, v, w \in W_{\mathrm{af}}^I$ and $x \in W_{\mathrm{af}}$,

$$(-1)^{\ell(w)-\ell(u)-\ell(v)}\phi_0(d_{uv}^w) \geqslant 0,$$
$$(-1)^{\ell(x)-\ell(u)}\phi_0(k_u^x) \geqslant 0.$$

By (5.4), the second statement implies the first.

The tables of $\phi_0(k_w)$ in Appendix A.3.3 support Conjecture 6.7.

6.3 Non-equivariant K-homology

One defines the non-equivariant K-cohomology $K^*(\mathrm{Gr}_G)$ by considering non-equivariant coherent sheaves in the natural way. We have $K^*(\mathrm{Gr}_G) = \bigoplus_{w \in W_{\mathrm{af}}^I} \mathbb{Z}\,[\mathcal{O}_{X_w^I}]_0$, where $[\mathcal{O}_{X_w^I}]_0$ denotes a non-equivariant class. The non-equivariant K-homology $K_*(\mathrm{Gr}_G)$, defined as the continuous $\mathbb{Z}$-dual to $K^*(\mathrm{Gr}_G)$, has Schubert basis $\{\xi_w^0 \mid w \in W_{\mathrm{af}}^I\}$. We have the following commutative diagram.

$$\begin{array}{ccc} K_T(\mathrm{Gr}_G) & \xrightarrow{\phi_0} & K_*(\mathrm{Gr}_G) \\ \updownarrow & & \updownarrow \\ \mathbb{L} & \xrightarrow[\phi_0]{} & \mathbb{L}_0 \end{array}$$

The subalgebra $\mathbb{L}_0$ is a Hopf algebra, with coproduct $\phi_0 \circ \Delta$. The following result generalizes [Lam08, Theorem 5.5] to K-homology.

Theorem 6.8. *There is a Hopf-isomorphism $k_0 : K_*(\mathrm{Gr}_G) \longrightarrow \mathbb{L}_0$ such that $k_0(\xi_w^0) = \phi_0(k_w)$.*

7. Grothendieck polynomials for the affine Grassmannian

In this section we specialize to affine type $A_{n-1}^{(1)}$ and $G = \mathrm{SL}_n(\mathbb{C})$. We first introduce elements $\mathbf{k}_i \in \mathbb{L}_0$ which, under a Hopf-algebra isomorphism $\mathbb{L}_0 \cong \Lambda_{(n)} := \mathbb{Z}[h_1, \dots, h_{n-1}]$ between the K-affine Fomin–Stanley algebra and a subspace of symmetric functions, correspond to the homogeneous symmetric functions h_i. For $w \in W_{\mathrm{af}}^I$, the image g_w of $\phi_0(k_w)$ in $\Lambda_{(n)}$ is the K-theoretic k-Schur function g_w which contains the k-Schur function (see [LLM03, LM07]) as the highest-degree term. The symmetric functions g_w are related to the affine stable Grothendieck polynomials $\{G_w \mid w \in W_{\mathrm{af}}^I\}$ of [Lam06] by duality.

7.1 Cyclically decreasing permutations and the elements $\mathbf{k}_i$

For $G = \mathrm{SL}_n$, we have $I = \{1, 2, \dots, n-1\}$ and $I_{\mathrm{af}} = \{0\} \cup I$. For $i \in I$ we wish to compute the elements $\phi_0(k_{\sigma_i}) \in \mathbb{L}_0$ where $\sigma_i = r_{i-1} r_{i-2} \cdots r_1 r_0 \in W_{\mathrm{af}}$.

A *cyclically decreasing* element $w \in W_{\mathrm{af}}$ is one that has a reduced decomposition $w = r_{i_1} r_{i_2} \cdots r_{i_N}$ such that the indices $i_1, \dots, i_N \in I_{\mathrm{af}}$ are all distinct and a reflection r_i never occurs somewhere to the left of a reflection r_{i+1}; here I_{af} is identified with $\mathbb{Z}/n\mathbb{Z}$, so that indices are computed mod n). One can show that w is cyclically decreasing if and only if all of its reduced decompositions have the above property. Since no non-commuting braid relations can occur, all the reduced words of w also have the same indices $i_1, \dots, i_N$.

For $i \in I$, let $\mathbf{k}_i \in \mathbb{K}_0$ be defined by

$$\mathbf{k}_i = \sum_w T_w \tag{7.1}$$

where w runs over the cyclically decreasing elements of W_{af} of length i. We set $\mathbf{k}_0 = 1$. These elements were considered in [Lam06].

We define coordinates for the weight lattice P of $\mathfrak{sl}_n$. Let $P \subset \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z} e_i$, with fundamental weights $\omega_i = e_1 + e_2 + \cdots + e_i$ and $\alpha_i = e_i - e_{i+1}$ for $i \in I$. For a subset $J \subset \{1, \dots, n\}$, let us write $e_J = \sum_{i \in J} e_i \in P$ for the 01-vector with 1s in the positions corresponding to elements of J. The e_J with $|J| = k$ form the set of weights for the kth fundamental representation of $\mathrm{SL}_n(\mathbb{C})$ with highest weight ω_k, which is multiplicity-free. We have $r_i \cdot e_J = e_{r_i \cdot J}$, where indices are taken mod n.

Lemma 7.1. *We have*

$$T_i \cdot e^{e_J} = \begin{cases} 0 & \text{if both } i, i+1 \in J \text{ or both } i, i+1 \notin J, \\ e^{e_{r_i \cdot J}} & \text{if } i \in J \text{ and } i+1 \notin J, \\ -e^{e_J} & \text{if } i \notin J \text{ and } i+1 \in J. \end{cases}$$

Let J and K be disjoint subsets of $\mathbb{Z}/n\mathbb{Z}$ such that $J \cup K \neq \mathbb{Z}/n\mathbb{Z}$. We write $S_{J,K} \cdot e^\lambda$ for the action of $\{r_j \mid j \in J\}$ and $\{T_k \mid k \in K\}$ on e^λ, where the operators act in cyclically decreasing order (for example, r_1 would act before T_2).

Lemma 7.2. *Let J and K be as above.*

(i) *If $|[0, k-1] \cap K| \geqslant 2$, then $S_{J,K} \cdot e^{\omega_k} = 0$.*

(ii) *Suppose $|[0, k-1] \cap K| = 1$. Let $a \in [0, k-1] \cap K$. Then $S_{J,K} \cdot e^{\omega_k} = 0$ unless $[0, a-1] \subset J$.*

(iii) *Suppose $|[0, k-1] \cap K| = 1$. Then $S_{J,K} \cdot e^{\omega_k} = 0$ if $[k, -1] \subset (J \cup K)$.*

(iv) *Suppose that* $S_{J,K} \cdot e^{\omega_k} \neq 0$ *and* $[k,-1] \cap K \neq \emptyset$. *For each* $a \in ([k,-1] \cap K)$, *we have* $[k,a] \subset (J \cup K)$.

(v) *Suppose that* $S_{J,K} \cdot e^{\omega_k} \neq 0$ *and* $[k,-1] \subset (J \cup K)$. *Then* $[0,k-1] \cap K = \emptyset$.

The following lemma is general (not just for affine type $A_{n-1}^{(1)}$).

Lemma 7.3. *Suppose* $a \in \mathbb{K}_0$. *If* $\lambda, \mu \in P$ *are such that* $\phi_0(ae^\lambda) = a$ *and* $\phi_0(ae^\mu) = a$, *then* $\phi_0(ae^{\lambda+\mu}) = a$.

Proof. Write $a(e^\lambda - 1) = \sum_w a_w T_w$ and $T_w e^\mu = \sum_v b_w^{v,\mu} T_v$ for $a_w, b_w^{v,\mu} \in R(T)$. We have $\phi_0(a_w) = 0$ for all w and $\phi_0(a(e^\mu - 1)) = 0$. Then

$$\begin{aligned}\phi_0(a(e^{\lambda+\mu} - 1)) &= \phi_0(a(e^\lambda - 1)e^\mu) + \phi_0(a(e^\mu - 1)) \\ &= \phi_0\left(\sum_w a_w T_w e^\mu\right) \\ &= \sum_v \phi_0\left(\sum_w a_w b_w^{v,\mu}\right) T_v = 0\end{aligned}$$

since $\phi_0 : R(T) \to \mathbb{Z}$ is a ring homomorphism. □

Proposition 7.4. *We have* $\mathbf{k}_i \in \mathbb{L}_0$.

Proof. By Lemma 7.3, it is enough to prove that $\phi_0(\mathbf{k}_i e^\lambda) = \mathbf{k}_i$ for λ being either a fundamental weight or the negative of a fundamental weight. We deal with the case where $\lambda = \omega_k$; negative fundamental weights are treated similarly.

When $\phi_0(\mathbf{k}_i(e^{\omega_k} - 1))$ is expanded in the T_w basis, only a term that involves cyclically decreasing w would occur with non-zero coefficient. Fix J. Let us show that $[T_J]\phi_0$ $(\mathbf{k}_i(e^{\omega_k} - 1)) = 0$, where T_J is the product of T_j with $j \in J$ in cyclically decreasing order and $[T_J]a$ denotes the coefficient of T_J in $a \in \mathbb{K}_0$. This is clear if $|J| = i$, by (2.6) and Lemma 7.1. So suppose that $|J| < i$. Then

$$[T_J]\phi_0(\mathbf{k}_i(e^{\omega_k} - 1)) = \sum_{K\,:\,|K|=i-|J| \text{ and } K\cap J=\emptyset} \phi_0(S_{J,K} e^{\omega_k}). \tag{7.2}$$

We will say that a subset K in the above sum is *good* if $S_{J,K} e^{\omega_k} \neq 0$. Let us define an involution ι on good subsets such that $\phi_0(S_{J,K} e^{\omega_k}) = -\phi_0(S_{J,\iota(K)} e^{\omega_k})$. By Lemma 7.2(i), we may write the set of good subsets as the disjoint union $S_0 \sqcup S_1$ where

$$S_0 = \{K : |K \cap [0,k-1]| = 0\} \quad \text{and} \quad S_1 = \{K : |K \cap [0,k-1]| = 1\}.$$

The involution satisfies $\iota(S_a) = S_{1-a}$.

Suppose that $K \in S_0$. Then $K \cap [k,-1] \neq \emptyset$ and we may set a to be the maximal element of $K \cap [k,-1]$. Let $\iota(K) = K \backslash \{a\} \cup \{b\}$ where $b \in [0,k-1]$ is minimal so that $j \notin J$ and $[0,j-1] \subset J$. One can check directly that $\iota(K)$ is also good. Using Lemma 7.1, one sees that $\iota(K)$ and K contribute different signs in (7.2).

Suppose that $K \in S_1$. Let a be the unique element in $K \cap [0,k-1]$. Using Lemma 7.2(iii), we pick $b \in [k,-1] \backslash (J \cup K)$ minimal in $[k,-1]$. Set $\iota(K) = K \backslash \{a\} \cup \{b\}$. Again, one can check that $\iota(K)$ is good and that K and $\iota(K)$ contribute different signs.

Finally, it follows from Lemma 7.2(ii) and (iv) that ι is an involution. □

Corollary 7.5. *For* $1 \leqslant i \leqslant n-1$, $\mathbf{k}_i = \phi_0(k_{\sigma_i})$.

Proof. By Proposition 7.4, $\mathbf{k}_i \in \mathbb{L}_0$, and by definition it has unique Grassmannian term T_{σ_i}. The result follows from Theorem 6.4. □

By forgetting equivariance, we obtain a K-homology Pieri rule for $K_*(\mathrm{Gr}_{\mathrm{SL}_n})$; see [LLMS, LM05] for the homological version.

COROLLARY 7.6. *For* $1 \leqslant i \leqslant n-1$, $\phi_0(d^w_{\sigma_i,v})$ *equals* $(-1)^{\ell(w)-\ell(v)-i}$ *times the number of cyclically decreasing elements* $x \in W_{\mathrm{af}}$ *with* $\ell(x) = i$ *and* $T_x T_v = \pm T_w$.

Proof. This follows from (5.4), (7.1) and Corollary 7.5. □

The K-cohomology Pieri rule is likely to be much more complicated; see [LLMS] for the cohomological version.

7.2 Coproduct of the $\mathbf{k}_i$

In this section we determine the coproduct $\phi_0(\Delta(\mathbf{k}_i))$ explicitly.

Let J and K be two subsets of $\mathbb{Z}/n\mathbb{Z}$ with total size less than $n-1$. We define a sequence of non-negative integers $\mathrm{cd}_{J,K} = (\mathrm{cd}(j) : j \in \mathbb{Z}/n\mathbb{Z})$ by

$$\mathrm{cd}(j) = \max_{0 \leqslant t \leqslant n-1} \{|J \cap [j-t, j)| + |K \cap [j-t, j)| - t\}.$$

(The intervals $[j-t, j)$ are to be considered as cyclic intervals.) It is then clear that $\mathrm{cd}(j+1) - \mathrm{cd}(j) \in \{-1, 0, 1\}$.

We note that $\mathrm{cd}(j) \geqslant 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

LEMMA 7.7. *Let J and K be two subsets of $\mathbb{Z}/n\mathbb{Z}$ with total size less than $n-1$.*

(i) *There exists j such that $\mathrm{cd}(j) = 0$ and $j \notin J \cup K$.*

(ii) cd *is the unique sequence such that* $\mathrm{cd}(j+1) - \mathrm{cd}(j) = |j \cap J| + |j \cap K| - 1$, *except when* $\mathrm{cd}(j) = 0$ *and* $j \notin (J \cup K)$.

Proof. To prove (i), suppose that no such j exists. Then $\mathrm{cd}(j+1) - \mathrm{cd}(j) = |j \cap J| + |j \cap K| - 1$ for each j. But $0 = (\mathrm{cd}(j) - \mathrm{cd}(j-1)) + \cdots + (\mathrm{cd}(j+1) - \mathrm{cd}(j))$, so this is impossible because $|J| + |K| \leqslant n-1$. Now we prove (ii). Everything except uniqueness is clear. Let cd' be any sequence with the asserted properties. The same calculation as in (i) shows that there is j' such that $\mathrm{cd}'(j') = 0 = \mathrm{cd}'(j'+1)$ and $j' \notin (J \cup K)$. By recursively calculating $\mathrm{cd}(j'+1)$ and $\mathrm{cd}'(j'+1)$, then $\mathrm{cd}(j'+2)$ and $\mathrm{cd}'(j'+2)$, and so on, we find that $\mathrm{cd}(j) \geqslant \mathrm{cd}'(j)$ for all j. But a symmetric argument shows that $\mathrm{cd}(j) \leqslant \mathrm{cd}'(j)$ for all j. □

Define $t(J,K) = (t_i : i \in \mathbb{Z}/n\mathbb{Z}) \in \{L, R, B, E\}^n$ as follows (here E stands for 'empty', L for 'left', R for 'right', and B for 'both'):

$$t_j = \begin{cases} E & \text{if } \mathrm{cd}(j) = 0 \text{ and } j \notin J \cup K, \\ L & \text{if } \mathrm{cd}(j) = 0 \text{ and } j \in J \backslash K, \\ R & \text{if } j \notin J \text{ and } (\mathrm{cd}(j) > 0 \text{ or } j \in K), \\ B & \text{otherwise.} \end{cases}$$

We say that two sequences cd and t are *compatible* if:

(a) $t_j \in \{E, L\}$ implies $\mathrm{cd}(j) = 0$;

(b) $\mathrm{cd}(j+1) - \mathrm{cd}(j) = 0$ if $t_j = L$;

(c) $\mathrm{cd}(j+1) - \mathrm{cd}(j) \in \{-1, 0\}$ if $t_j = R$;

(d) $\mathrm{cd}(j+1) - \mathrm{cd}(j) \in \{0,1\}$ if $t_j = B$;

(e) $t_j = E$ for some $j \in [0, n-1]$.

Define the support of (cd, t) to be $\{j \mid t_j \neq E\}$.

Lemma 7.8. *The map $(J,K) \mapsto (\mathrm{cd}_{J,K}, t(J,K))$ is a bijection between pairs of subsets of $\mathbb{Z}/n\mathbb{Z}$ with total size $k < n$ and pairs of compatible sequences with support of size k.*

Proof. It is easy to see that $(\mathrm{cd}_{J,K}, t(J,K))$ is compatible with support of the correct size. We first check that the pair of sequences determines J and K. By itself, $t(J,K)$ completely determines J: we have $j \in J$ if and only if $t_j \in \{L, B\}$. Also, $j \in K$ if and only if either $t_j = R$ and $\mathrm{cd}(j+1) = \mathrm{cd}(j)$ or $t_j = B$ and $\mathrm{cd}(j+1) = \mathrm{cd}(j) + 1$. Thus $(\mathrm{cd}_{J,K}, t(J,K))$ determines (J,K).

Conversely, given compatible (cd, t), we recursively construct J and K by starting at some value j such that $t_j = E$. For such a value we have $j \notin J \cup K$. We then decide whether $j+1 \in J$ and/or $j+1 \in K$, and so on. By construction, we obtain two subsets J and K such that $\mathrm{cd}(j+1) - \mathrm{cd}(j) = |j \cap J| + |j \cap K| - 1$, unless $\mathrm{cd}(j) = 0 = \mathrm{cd}(j+1)$ and $j \notin J \cup K$. By Lemma 7.7, we have $\mathrm{cd}_{J,K} = \mathrm{cd}$. Using compatibility, one can check that the size of the support of (cd, t) is equal to $|J| + |K|$. But then it follows that $t(J,K) = t$. □

Proposition 7.9. *We have $\phi_0(\Delta(\mathbf{k}_r)) = \sum_{0 \leqslant j \leqslant r} \mathbf{k}_j \otimes \mathbf{k}_{r-j}$.*

Proof. Our proof follows the strategy in [Lam08, § 7.2]. Let $J = \{i_1, \ldots, i_r\} \subset \mathbb{Z}/n\mathbb{Z}$. Using (2.16), we calculate $\phi_0(\Delta(T_J))$ by expanding

$$D = \phi_0(\Delta(T_{i_1}) \cdot \Delta(T_{i_2}) \cdots \Delta(T_{i_\ell})),$$

where $r_{i_1} \cdots r_{i_\ell}$ is a cyclically decreasing reduced expression and $\cdot$ means the 'componentwise' product on $\Delta(\mathbb{K})$ of (2.17).

Let us expand this product by picking, for each component, one of the three terms of (2.16). As usual, we write $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$ for any cyclic interval $[i,j]$. We first note that

$$\begin{aligned} T_j(1 - e^{\alpha_{i,j}}) &= (1 - e^{\alpha_{i,j+1}})T_j + \frac{1 - e^{\alpha_{i,j+1}} - (1 - e^{\alpha_{i,j}})}{1 - e^{\alpha_j}} \\ &= (1 - e^{\alpha_{i,j+1}})T_j + e^{\alpha_{i,j}}. \end{aligned}$$

Because of the cyclically decreasing condition, whenever the above calculation is encountered, the coefficient $e^{\alpha_{i,j}}$ will always commute with any T_i which occurs to the left.

We shall now show by induction that the only terms in the expansion of $\Delta(T_{i_k}) \cdots \Delta(T_{i_\ell})$ that contribute to D look like

$$T_v \otimes \prod_{i \in S} (1 - e^{\alpha_{i,i_k+1}}) q T_w \tag{7.3}$$

where:

(i) either S is empty and $\prod_{i \in S}(1 - e^{\alpha_{i,i_k+1}}) = 1$, or $S \subset \{i_k, i_{k+1}, \ldots, i_\ell\}$;

(ii) $q \in R(T)$ commutes with $r_{i_1}, \ldots, r_{i_{k-1}}$ and satisfies $\phi_0(q) = 1$.

Such a term contributes nothing to D if $|S| > 0$ and $i_k \notin \{i_1, \ldots, i_{k-1}\}$. To prove the inductive step, we assume that $i_{k-1} = i_k + 1$ and calculate (using (2.5))

$$T_{i_k+1} \prod_{i \in S} (1 - e^{\alpha_{i,i_k+1}}) q = \prod_{i \in S} (1 - e^{\alpha_{i,i_k+2}}) q T_{i_k+1} + \prod_{i \in S'} (1 - e^{\alpha_{i,i_k+2}}) + q', \tag{7.4}$$

where $S' \subset S$ and $q' \in R(T)$ commutes with $r_{i_1}, \ldots, r_{i_{k-2}}$ and satisfies $\phi_0(q') = 0$. Clearly, the term involving q' contributes nothing to D, and the first two terms lead to expressions of the form (7.3).

Given a choice of one of the three terms in (2.16), we define a sequence t_j by:

(a) $t_j = E$ if $j \notin J$;

(b) $t_j = L$ if we pick $T_j \otimes 1$;

(c) $t_j = R$ if we pick $1 \otimes T_j$;

(d) $t_j = B$ if we pick $T_j \otimes (1 - e^{\alpha_j})T_j$.

Furthermore, let us make a choice of one of the two terms in (7.4), whenever we have such a choice. At each step of our calculation we are looking at a term of the form (7.3). We set $\mathrm{cd}(j)$ to be the size of S in the term just before $\Delta(T_j)$ is applied. If $j \notin J$, then $\mathrm{cd}(j) = 0$. If this entire process produces a non-zero term of D, then (cd, t) is a compatible sequence: the sequence cd 'wraps around' properly because eventually the coefficient $\prod_{i \in S}(1 - e^{\alpha_{i,i_k+1}})$ has to equal 1, otherwise it will vanish when ϕ_0 is applied. Conversely, a compatible pair (cd, t) with support equal to J always arises in this fashion.

By Lemma 7.8, there is a bijection between compatible pairs (cd, t) with support of size r and pairs of subsets (J, K) with total size equal to r. It is easy to check that the term in D corresponding to (cd, t) is exactly $T_J \otimes T_K$. □

7.3 Symmetric function realizations

Let $\Lambda = \bigoplus_\lambda \mathbb{Z} m_\lambda$ be the ring of symmetric functions over $\mathbb{Z}$, where m_λ is the monomial symmetric function [Mac95] and $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell > 0)$ runs over all partitions. Let $\hat{\Lambda} = \prod_\lambda \mathbb{Z} m_\lambda$ be the graded completion of Λ. Let $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ denote the size of a partition.

Let $\Lambda^{(n)} = \Lambda / \langle m_\lambda \mid \lambda_1 \geqslant n \rangle$ denote the quotient by the ideal generated by monomial symmetric functions labeled by partitions with first part greater than n. We write $\hat{\Lambda}^{(n)}$ for the graded completion of $\Lambda^{(n)}$. Now let $\Lambda_{(n)} = \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \subset \Lambda$ denote the subalgebra generated by the first $n-1$ homogeneous symmetric functions. Both $\Lambda_{(n)}$ and $\hat{\Lambda}^{(n)}$ are Hopf algebras.

The *Hall inner product* $\langle \cdot, \cdot \rangle : \Lambda \times_{\mathbb{Z}} \Lambda \to \mathbb{Z}$ extends by linearity with respect to infinite graded linear combinations to a pairing $\langle \cdot, \cdot \rangle : \Lambda \times_{\mathbb{Z}} \hat{\Lambda} \to \mathbb{Z}$, which in turn descends to a pairing $\Lambda_{(n)} \times_{\mathbb{Z}} \Lambda^{(n)} \to \mathbb{Z}$. This pairing expresses $\Lambda_{(n)}$ as the continuous (Hopf-)dual of $\hat{\Lambda}^{(n)}$ and $\hat{\Lambda}^{(n)}$ as the graded completion of the graded (Hopf-)dual of $\Lambda_{(n)}$. For short, we will just say that $\Lambda_{(n)}$ and $\hat{\Lambda}^{(n)}$ are dual. The basis $\{h_\lambda \mid \lambda_1 < n\} \subset \Lambda_{(n)}$ and 'basis' $\{m_\lambda \mid \lambda_1 < n\} \subset \hat{\Lambda}^{(n)}$ are dual under the Hall inner product.

7.4 Affine stable Grothendieck polynomials

The *affine stable Grothendieck polynomials* $G_v(x_1, x_2, \ldots)$ for $v \in W_{\mathrm{af}}$ are the formal power series defined by the identity [Lam06]

$$\prod_{i \geqslant 1} \sum_{j=0}^{n-1} (x_i^j \mathbf{k}_j) = \sum_{v \in W_{\mathrm{af}}} G_v(x_1, x_2, \ldots) T_v, \tag{7.5}$$

where the x_i are indeterminates that commute with the elements of $\mathbb{K}_0$ and $\mathbf{k}_j \in \mathbb{K}_0$ are the elements defined in (7.1). Alternatively, for a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, the coefficient of $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$ in $G_v(x)$ is equal to the coefficient of T_v in $\mathbf{k}_{\alpha_1} \mathbf{k}_{\alpha_2} \cdots \mathbf{k}_{\alpha_\ell}$. It is clear that

$G_v(x)$ is a sum of monomials such that no variable occurs with degree more than $n-1$ in any monomial. Examples of the $G_v(x)$ are given in Appendix A.3.6.

The following result was proved directly in [Lam06, Theorem 44].

Proposition 7.10. *For each $v \in W_{\mathrm{af}}$, we have $G_v(x) \in \hat{\Lambda}^{(n)}$.*

Proof. By Proposition 7.4 and Corollary 6.5, the $\mathbf{k}_i$ commute. This implies that $G_v(x)$ is a symmetric function. In addition, all monomial symmetric functions m_λ which occur with non-zero coefficient in $G_v(x)$ satisfy $\lambda_1 < n$, so $G_v(x)$ can be naturally identified with its image in $\hat{\Lambda}^{(n)}$. □

The next result follows from (2.3).

Lemma 7.11. *The graded components of $G_v(x)$ are alternating; that is, the coefficient of m_λ in $G_v(x)$ has sign equal to that of $(-1)^{|\lambda|-\ell(v)}$.*

In [Lam06], for each $v \in W_{\mathrm{af}}$, a homogeneous symmetric function $F_v(x)$, the *affine Stanley symmetric function*, is defined. The next result follows by inspection.

Lemma 7.12. *Let $v \in W_{\mathrm{af}}^I$. The lowest-degree component (of degree $\ell(v)$) of $G_v(x)$ is equal to $F_v(x)$.*

(Readers not familiar with affine Stanley symmetric functions may take this as the definition of $F_v(x)$.) The $F_v(x)$, $v \in W_{\mathrm{af}}^I$, were called *affine Schur functions* in [Lam06] and are equivalent to the dual k-Schur functions of [LM07].

Proposition 7.13. *The set $\{G_v(x) \mid v \in W_{\mathrm{af}}^I\}$ is a 'basis' of $\hat{\Lambda}^{(n)}$. In other words, $\hat{\Lambda}^{(n)} = \prod_{v \in W_{\mathrm{af}}^I} \mathbb{Z}\, G_v(x)$.*

Proof. This follows from the fact that $\{F_v(x) \mid v \in W_{\mathrm{af}}^I\}$ is a basis of $\Lambda^{(n)}$; see [Lam06, LM07]. □

Remark 7.1. Suppose $w \in W_{\mathrm{af}}$ is such that some (or, equivalently, every) reduced expression for w does not involve all of the simple generators $r_0, r_1, \ldots, r_{n-1}$. It then follows from comparing the definitions that the stable affine Grothendieck polynomial $G_w(x)$ is equal to the usual stable Grothendieck polynomial [FK94] labeled by $u \in W = S_n$, where u is obtained from w by cyclically rotating the indices until r_0 is not present.

Remark 7.2. There is a bijection between $v \in W_{\mathrm{af}}^I$ and $(n-1)$-bounded partitions $\{\lambda \mid \lambda_1 < n\}$; see [LM05, Lam06]. The partition λ associated to v can be obtained from the exponents of the dominant monomial term $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ in $F_v(x_1, x_2, \ldots)$. We may thus relabel $\{G_v(x) \mid v \in W_{\mathrm{af}}^I\}$ as $\{G_\lambda^{(k)}(x) \mid \lambda_1 < n\}$, where $k = n-1$ (owing to historical reasons). A table showing this correspondence is given in Appendix A.3.1. Remark 7.1 implies that $G_\lambda^{(k)}(x) = G_\lambda(x)$ whenever the largest hook length of λ is less than or equal to k, where $G_\lambda(x)$ is the stable Grothendieck polynomial labeled by partitions and studied by Buch in [Buc02].

7.5 K-theoretic k-Schur functions

Since $\{G_v(x) \mid v \in W_{\mathrm{af}}^I\}$ is a 'basis' of $\hat{\Lambda}^{(n)}$, there is a dual basis $\{g_v(x) \mid v \in W_{\mathrm{af}}^I\}$ of $\Lambda_{(n)}$. This definition of $g_v(x)$ has been stated previously by Lam. We call the symmetric functions $g_v(x)$ *affine dual stable Grothendieck polynomials* or K-theoretic k-Schur functions. Examples of the $g_v(x)$ are given in Appendix A.3.4.

The proof of the following result is standard (see, for example, [Sta01, Lemma 7.9.2]).

LEMMA 7.14. *We have*

$$\sum_{\lambda_1<n} h_\lambda(x)\, m_\lambda(y) = \sum_{v\in W_{\mathrm{af}}^I} g_v(x)\, G_v(y).$$

Let $k=n-1$. The *k-Schur functions* $\{s_v^{(k)}(x) \mid v\in W_{\mathrm{af}}^I\}$ (see [LLM03, LM07]) form the basis of $\Lambda_{(n)}$ dual to $\{F_v(x)\mid v\in W_{\mathrm{af}}^I\}$, and are usually labeled by the k-bounded partitions $\{\lambda \mid \lambda_1<n\}$. (The k-Schur functions originally defined in [LLM03] depend on an additional parameter t, and setting $t=1$ conjecturally gives the k-Schur functions of [LM07], which are the ones used here.)

LEMMA 7.15. *Let $v\in W_{\mathrm{af}}^I$. Then the highest-degree homogeneous component of $g_v(x)$ is equal to the k-Schur function $s_v^{(k)}(x)$.*

Proof. We prove this by induction on $\ell(v)$. The base case is clear: $G_{\mathrm{id}}(x)=F_{\mathrm{id}}(x)=1=g_{\mathrm{id}}(x)=s_{\mathrm{id}}^{(k)}(x)$. Suppose the claim has been proven for all w satisfying $\ell(w)<\ell$, and let $\ell(v)=\ell$. One then checks that the symmetric function

$$g_v(x) = s_v^{(k)}(x) - \sum_{w\colon \ell(w)<\ell} \langle s_v^{(k)}(x), G_w(x)\rangle g_w(x)$$

is a solution to the system of equations

$$\langle g_v(x), G_u(x)\rangle = \delta_{vu} \quad \text{for all } u\in W_{\mathrm{af}}^I. \qquad \square$$

Remark 7.3. Relabel $\{g_v(x)\mid v\in W_{\mathrm{af}}^I\}$ as $\{g_\lambda^{(k)}(x)\mid \lambda_1\leqslant k\}$, as in Remark 7.2. Since $\lim_{k\to\infty} G_\lambda^{(k)}(x)=G_\lambda(x)$, it follows that $\lim_{k\to\infty} g_\lambda^{(k)}(x)=g_\lambda(x)$, where the $g_\lambda(x)$ are the dual affine stable Grothendieck polynomials studied in [LP07, Len00].

7.6 Non-commutative K-theoretic k-Schur functions

Define $\varphi:\Lambda_{(n)}\to\mathbb{L}_0$ by $h_i\mapsto\mathbf{k}_i$. This map is well-defined since the h_i are algebraically independent and $\mathbb{L}_0$ is commutative. The *non-commutative K-theoretic k-Schur functions* are the elements $\{\varphi(g_v)\mid v\in W_{\mathrm{af}}^I\}\subset\mathbb{L}_0$.

PROPOSITION 7.16. *Let $w\in W_{\mathrm{af}}$ and $v\in W_{\mathrm{af}}^I$. The coefficient of T_w in $\varphi(g_v)$ is equal to the coefficient of $G_v(x)$ in $G_w(x)$ when the latter is expanded in terms of $\{G_u(x)\mid u\in W_{\mathrm{af}}^I\}$.*

Proof. Applying φ to Lemma 7.14 and comparing with (7.5), we have

$$\sum_{v\in W_{\mathrm{af}}^I} G_v(y)\varphi(g_v) = \sum_{w\in W_{\mathrm{af}}} G_w(y)\, T_w.$$

Now take the coefficient of T_w on both sides. $\square$

7.7 Grothendieck polynomials for the affine Grassmannian

The following is our main theorem.

THEOREM 7.17.

(i) *The map $\varphi:\Lambda_{(n)}\to\mathbb{L}_0$ is a Hopf-isomorphism, sending g_v to $\phi_0(k_v)$ for $v\in W_{\mathrm{af}}^I$.*

(ii) *We have a Hopf-algebra isomorphism $k_0^{-1}\circ\varphi:\Lambda_{(n)}\to K_*(\mathrm{Gr}_{\mathrm{SL}_n})$, sending g_v to ξ_v^0 for $v\in W_{\mathrm{af}}^I$.*

(iii) *There is a dual Hopf-algebra isomorphism* $K^*(\mathrm{Gr}_{\mathrm{SL}_n}) \cong \hat{\Lambda}^{(n)}$*, sending* $[\mathcal{O}_{X_v^I}]_0$ *to* $G_v(x)$ *for* $v \in W_{\mathrm{af}}^I$.

(iv) *The following diagram commutes.*

$$\begin{array}{ccc} K_*(\mathrm{Gr}_{\mathrm{SL}_n}) \times K^*(\mathrm{Gr}_{\mathrm{SL}_n}) & \longrightarrow & \mathbb{Z} \\ \updownarrow & & \downarrow \mathrm{id} \\ \Lambda_{(n)} \times \hat{\Lambda}^{(n)} & \longrightarrow & \mathbb{Z} \end{array}$$

Proof. Given the definitions and Theorem 6.8, all the statements follow from the first one. By Theorem 6.4 and Propositions 7.13 and 7.16, we deduce that $\varphi(g_v) = \phi_0(k_v)$. It follows that φ is an isomorphism. Since $\Delta(h_i) = \sum_{0 \leqslant j \leqslant i} h_j \otimes h_{i-j}$ in $\Lambda_{(n)}$, it follows from Proposition 7.9 that φ is a Hopf-morphism. □

Corollary 7.18. *For* $1 \leqslant r \leqslant n-1$*, we have* $g_{\sigma_r}(x) = h_r(x)$.

Recall the map $r^* : K^T(X_{\mathrm{af}}) \to K^T(\mathrm{Gr}_G)$ defined in § 5.1. We use $r_0^* : K^*(X_{\mathrm{af}}) \to K^*(\mathrm{Gr}_{\mathrm{SL}_n})$ to denote the evaluation of r^* at zero.

Theorem 7.19. *The image of* $G_w(x)$ *under the isomorphism* $\hat{\Lambda}^{(n)} \cong K^*(\mathrm{Gr}_{\mathrm{SL}_n})$ *is equal to* $r_0^*([\mathcal{O}_{X_w}]_0)$.

Proof. As observed previously, the map ϖ of Lemma 4.6 is related to r^* via the isomorphisms of Theorem 4.3. By (5.1), $\langle k(\xi_v), \psi^w \rangle = \langle \xi_v, \varpi(\psi^w) \rangle$. It follows that the coefficient of T_w in k_v is equal to the coefficient of $[\mathcal{O}_{X_v^I}]$ in $r^*([\mathcal{O}_{X_w}])$. Applying ϕ_0 to these coefficients and comparing with Proposition 7.16 gives the result. □

7.8 Conjectural properties

In this section we list conjectural properties of the symmetric functions $g_w(x)$ and $G_w(x)$. When $w \in W_{\mathrm{af}}^I$, we will use partitions to label these symmetric functions; see Remark 7.2. Recall also that $k = n-1$.

Conjecture 7.20. The basis $\{g_\lambda^{(k)}\}$ of $\Lambda_{(n)}$ has the following properties.

(i) Each $g_\lambda^{(k)}$ is a positive integer (necessarily finite) sum of k-Schur functions. (By Lemma 7.15, the top homogeneous component of $g_\lambda^{(k)}$ is the k-Schur function $s_\lambda^{(k)}$.)

(ii) The coproduct structure constants $c_\lambda^{\mu\nu}$ in $\Delta(g_\lambda^{(k)}) = \sum_{\mu,\nu} c_\lambda^{\mu\nu} g_\mu^{(k)} \otimes g_\nu^{(k)}$ are alternating integers, that is, $(-1)^{|\lambda|-|\nu|-|\mu|} c_\lambda^{\mu\nu} \in \mathbb{Z}_{\geqslant 0}$. Furthermore, $c_\lambda^{\mu\nu} = 0$ unless $|\mu| + |\nu| \leqslant |\lambda|$.

(iii) The coefficients in the expansion $g_\lambda^{(k)} = \sum_\mu a_\lambda^\mu \, g_\mu^{(k+1)}$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|} a_\lambda^\mu \in \mathbb{Z}_{\geqslant 0}$.

Conjecture 7.20(1) has been checked for $n = 2, 3, 4$ and 5 for $|\lambda| \leqslant 8$ using the software package Sage [Sag]; see also the tables in Appendix A.3.4. Data confirming Conjecture 7.20(ii) can be found in Appendix A.3.5. Conjecture 7.20(iii) has been checked for $n = 2, 3$ and 4 for $|\lambda| \leqslant 8$ using Sage. According to Conjecture 6.7, the product structure constants for $\{g_\lambda^{(k)}\}$ should be alternating.

CONJECTURE 7.21.

(i) Every affine stable Grothendieck polynomial G_w for $w \in W_{\mathrm{af}}$ is a *finite* alternating linear combination of $\{G_\lambda^{(k)}\}$.

(ii) Every $G_\lambda^{(k)}$ is an *alternating* integer linear combination of the affine Schur functions $\{F_\mu^{(k)}\}$.

(iii) The structure constants in the product $G_\mu^{(k)}\, G_\nu^{(k)} = \sum_\lambda c_\lambda^{\mu\nu}\, G_\lambda^{(k)}$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|-|\nu|} c_\lambda^{\mu\nu} \in \mathbb{Z}_{\geqslant 0}$. Furthermore, $c_\lambda^{\mu\nu} = 0$ unless $|\mu| + |\nu| \leqslant |\lambda|$.

(iv) The coefficients in the expansion $G_\mu^{(k+1)} = \sum_\lambda a_\lambda^\mu\, G_\lambda^{(k)}$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|} a_\lambda^\mu \in \mathbb{Z}_{\geqslant 0}$.

By Proposition 7.16, the 'alternating' part of Conjecture 7.21(i) is implied by Conjecture 6.7. Evidence for Conjecture 7.21(ii) is provided in the table of Appendix A.3.6. Conjecture 7.21(ii) is related to Conjecture 7.20(i) via a matrix inverse. Conjecture 7.21(iii) is equivalent to Conjecture 7.20(ii); indeed, the two sets of structure constants are identical. Conjecture 7.21(iv) is equivalent to Conjecture 7.20(iii).

Remark 7.4. The factorization of affine Grassmannian homology Schubert classes as described in [Mag] (see also [Lam08, LM07]) appears also to hold in some form in K-homology. Suppose that $w \in W_{\mathrm{af}}^I$ has a length-additive factorization $w = vu$ where $u \in W_{\mathrm{af}}^I$ is equal, modulo length-zero elements, to the translation $t_{-\omega_i^\vee}$ by a negative fundamental coweight in the extended affine Weyl group [Mag] or, equivalently, that the partition λ corresponding to u is a rectangle of the form $\ell \times (k-\ell)$. Then it appears that g_w is a multiple of g_u in $\Lambda_{(n)}$.

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Appendix A. The affine NilHecke ring and tables

A.1 The (cohomological) affine NilHecke ring

A summary of the (notational) correspondence between (co)homology and K-(co)homology is given in Table A1.

Some of our notation differs from that in [Lam08].

We now recall the affine NilHecke ring $\mathbb{A}$. Let $S = \mathrm{Sym}(P)$ where P is the weight lattice of the finite-dimensional group G. Then W_{af} acts on P (and therefore on $S \cong H^T(\mathrm{pt})$) by the level-zero action. The affine NilCoxeter algebra $\mathbb{A}_0$ is the ring with generators $\{A_i \mid i \in I_{\mathrm{af}}\}$ and relations

$$A_i^2 = 0 \quad \text{and} \quad \underbrace{A_i A_j \cdots}_{m_{ij}\ \text{times}} = \underbrace{A_j A_i \cdots}_{m_{ij}\ \text{times}}.$$

Define A_w in the obvious way and define the NilCoxeter algebra by $\mathbb{A}_0 = \bigoplus_{w \in W_{\mathrm{af}}} \mathbb{Z} A_w$. Then $\mathbb{A}_0$ acts on S by

$$A_i \cdot \lambda = \langle \alpha_i^\vee, \lambda \rangle,$$
$$A_i \cdot (ss') = (r_i \cdot s) A_i \cdot s' + (A_i \cdot s) s'$$

for $i \in I_{\mathrm{af}}$, $\lambda \in P$ and $s, s' \in S$.

Table A1. Terminology.

(Co)homology	K-(co)homology	Terminology
A_i	T_i	(K-)NilHecke generators
$\mathbb{A} = \bigoplus_w SA_w$	$\mathbb{K} = \bigoplus_w R(T)T_w$	(K-)NilHecke ring
$\mathbb{B} = Z_{\mathbb{A}}(S)$	$\mathbb{L} = Z_{\mathbb{K}}(R(T))$	Peterson's subalgebra
$\mathbb{B}_0 = \phi_0(\mathbb{B})$	$\mathbb{L}_0 = \phi_0(\mathbb{L})$	Affine Fomin–Stanley subalgebra
$\{j_w\} \subset \mathbb{B}$	$\{k_w\} \subset \mathbb{L}$	Schubert basis
$s_w^{(k)}(x)$	$g_w(x)$	(K-theoretic) k-Schur functions
$F_w(x)$	$G_w(x)$	Affine Stanley symmetric functions/stable affine Grothendieck polynomials

The affine Kostant–Kumar NilHecke ring $\mathbb{A}$ (see [Pet97]) is the smash product of $\mathbb{A}_0$ and S. It has relations

$$A_i s = (r_i \cdot s)A_i + (A_i \cdot s)$$

for $i \in I_{\mathrm{af}}$ and $s \in S$. Then

$$\mathbb{A} = \bigoplus_{w \in W_{\mathrm{af}}} SA_w.$$

In $\mathbb{A}$ we have $r_i = 1 - \alpha_i A_i$, and $\mathbb{A}$ acts on $\mathrm{Fun}(W, S)$ by

$$(a \cdot \xi)(w) = \xi(wa),$$

viewing $\xi \in \mathrm{Fun}(W, S)$ as an element of $\mathrm{Hom}_Q(\mathbb{A}_Q, Q)$ (the left Q-module homomorphisms) where $Q = \mathrm{Frac}(S)$.

Lemma A.1 [KK86]. *In $\mathbb{A}$, we have*

$$A_w \lambda = (w \cdot \lambda)A_w + \sum_{v = wr_\alpha \lessdot w} \langle \alpha^\vee, \lambda \rangle A_v.$$

Let $\phi_0 : S \to \mathbb{Z}$ be defined by evaluation at zero. Let $\phi_0 : \mathbb{A} \to \mathbb{A}_0$ be the map defined by $\phi_0(\sum_w a_w A_w) = \sum_w \phi_0(a_w)A_w$ for $a_w \in S$. Let $\mathbb{B} = Z_{\mathbb{A}}(S)$ be the Peterson subalgebra [Pet97], the centralizer subalgebra of S in $\mathbb{A}$, and let

$$\mathbb{B}_0 = \{b \in \mathbb{B} \mid \phi_0(bs) = \phi_0(s)b \text{ for all } s \in S\}$$

be the Fomin–Stanley subalgebra [Lam08].

Theorem A.2.

(i) *[Pet97] For each $w \in W_{\mathrm{af}}^I$, there is a unique element $j_w \in \mathbb{B}$ such that*

$$j_w \in A_w + \bigoplus_{v \in W \setminus W_{\mathrm{af}}^I} SA_v,$$

$$\mathbb{B} = \bigoplus_{w \in W_{\mathrm{af}}^I} S\, j_w.$$

(ii) *[Lam08]*

$$\mathbb{B}_0 = \phi_0(\mathbb{B}).$$

(iii) *For each* $w \in W_{\mathrm{af}}^I$, $\phi_0(j_w)$ *is the unique element of* $\mathbb{B}_0$ *such that*

$$\phi_0(j_w) \in A_w + \bigoplus_{v \in W \setminus W_{\mathrm{af}}^I} \mathbb{Z} A_v,$$

$$\mathbb{B}_0 = \bigoplus_{w \in W_{\mathrm{af}}^I} \mathbb{Z}\, \phi_0(j_w).$$

Compare these results with the K-theoretic analogues of Theorems 5.4 and 6.4.

For $G = \mathrm{SL}_{k+1}$, the element $\phi_0(j_w)$ is called a non-commutative k-Schur function [Lam08].

A.2 Comparison with the fixed-point functions of [KK90]

A.2.1 *Möbius inversion for Bruhat order.* The Möbius function for the Bruhat order on W is

$$(v, w) \mapsto (-1)^{\ell(w)-\ell(v)} \chi(v \leqslant w)$$

where $\chi(P) = 1$ if P is true and $\chi(P) = 0$ if P is false; see [Deo77]. In other words, let M be the $W \times W$ incidence matrix $M_{vw} = \chi(v \leqslant w)$ of the Bruhat order, and let N be the Möbius matrix $N_{vw} = \chi(v \leqslant w)(-1)^{\ell(w)-\ell(v)}$. Then M and N are inverse to each other:

$$\sum_{\substack{v \\ u \leqslant v \leqslant w}} (-1)^{\ell(w)-\ell(v)} = \delta_{uw} = \sum_{\substack{v \\ u \leqslant v \leqslant w}} (-1)^{\ell(v)-\ell(u)}. \tag{A1}$$

A.2.2 *Kostant and Kumar functions.* We now return to the (K-theoretic) notation of § 2. The following lemma is standard.

LEMMA A.3.

$$y_w = \sum_{v \leqslant w} T_v. \tag{A2}$$

For $v \in W$ define $\psi^v_{KK} \in \mathrm{Fun}(W, Q(T))$ by[2]

$$\psi^v_{KK}(y_w) = \delta_{vw}.$$

This is equivalent to

$$w = \sum_v \psi^v_{KK}(w) y_v. \tag{A3}$$

By (A3) and (A2) we have

$$w = \sum_v \psi^v_{KK}(w) \sum_{u \leqslant v} T_u = \sum_u T_u \sum_{v \geqslant u} \psi^v_{KK}(w).$$

By Proposition 2.4, (2.7) and (A1) we have

$$\psi^u = \sum_{v \geqslant u} \psi^v_{KK},$$

$$\psi^u_{KK} = \sum_{v \geqslant u} (-1)^{\ell(v)-\ell(u)} \psi^v.$$

[2] The functions denoted by $\psi^v(w)$ in [KK90] are the same as the functions that we denote by $\psi^{v^{-1}}_{KK}(w^{-1})$.

Recall the definition of η from Remark 2.4. One can show the following. Let ρ be the sum of fundamental weights.

Lemma A.4. *For all* $v, w \in W$,

$$\psi^v_{KK}(w) = (-1)^{\ell(v)} e^{\rho - w\rho} \eta(\psi^v(w)).$$

Remark A.1. Let $\partial X_v = X_v \backslash X_v^o$ be the boundary of the Schubert cell X_v^o in the Schubert variety X_v. Then there is an exact sequence

$$0 \to I_{\partial X_v \subset X_v} \to \mathcal{O}_{X_v} \to \mathcal{O}_{\partial X_v} \to 0.$$

Since $[\mathcal{O}_{X_u}] \mapsto \psi^u$ under the isomorphism $K^T(X) \to \Psi$, $[I_{\partial X_v \subset X_v}] \mapsto \psi^v_{KK}$.

A.3 Tables

A.3.1 *Table on Grassmannians versus k-bounded partitions.* We list the correspondence between reduced words for Grassmannian elements and k-bounded partitions, where $k = n - 1$.

n	k-bounded partition	$w \in W_{\mathrm{af}}^I$
2	1	0
	11	10
	111	010
	1111	1010
	11111	01010
3	1	0
	2	10
	11	20
	21	210
	111	120
	22	0210
	211	2120
	1111	0120
	221	10210
	2111	02120
	11111	20120

n	k-bounded partition	$w \in W_{\mathrm{af}}^I$
4	1	0
	2	10
	11	30
	3	210
	21	130
	111	230
	31	3210
	22	0130
	211	2130
	1111	1230
	32	03210
	311	32130
	221	20130
	2111	21230
	11111	01230

A.3.2 $\hat{SL}_2$. Set $\alpha = \alpha_1 = -\alpha_0$. We have $t_\alpha = r_0 r_1$ and $t_{-\alpha} = r_1 r_0$. Indexing T_w and k_w by reduced words, we have

$$t_\alpha = (1 - e^{-\alpha})^2 T_{01} + (1 - e^{-\alpha})(T_0 + T_1) + 1,$$
$$t_{-\alpha} = (1 - e^{\alpha})^2 T_{10} + (1 - e^{\alpha})(T_0 + T_1) + 1,$$
$$k_\emptyset = 1,$$
$$k_0 = T_0 + T_1 + (1 - e^{-\alpha}) T_{01},$$
$$k_{10} = T_{10} + e^{-\alpha} T_{01}.$$

So

$$\phi_0(k_\emptyset) = 1,$$
$$\phi_0(k_0) = T_0 + T_1,$$
$$\phi_0(k_{10}) = T_{10} + T_{01}.$$

In general,

$$\phi_0(k_{\sigma_r}) = T_{\sigma_r} + T_{\sigma_{-r}},$$

where σ_r are the elements in (4.4).

A.3.3 *Table of* $\phi_0(k_w)$. We index T_w by reduced words.

n	w	$\phi_0(k_w)$
3	$\emptyset$	1
	0	$T_0 + T_1 + T_2$
	10	$T_{10} + T_{02} + T_{21}$
	20	$T_{20} + T_{01} + T_{12}$
	210	$T_{210} + T_{020} + T_{021} + T_{101} + T_{102} + T_{212}$
	120	$T_{120} + T_{201} + T_{202} + T_{010} + T_{012} + T_{121}$
	0210	$T_{0210} + T_{1021} + T_{2102}$
	1210	$T_{1210} + T_{0201} + T_{0212} + T_{1020} + T_{1012} + T_{2101} - T_{020} - T_{101} - T_{212}$
	0120	$T_{0120} + T_{2012} + T_{1201}$
4	$\emptyset$	1
	0	$T_0 + T_1 + T_2 + T_3$
	10	$T_{10} + T_{21} + T_{32} + T_{03} + T_{02} + T_{13}$
	30	$T_{30} + T_{01} + T_{12} + T_{23} + T_{02} + T_{13}$
	210	$T_{210} + T_{103} + T_{032} + T_{321}$
	310	$T_{130} + T_{132} + T_{021} + T_{023} + T_{030} + T_{031} + T_{320} + T_{323} + T_{213} + T_{212} + T_{101} + T_{102} - T_{02} - T_{13}$
	230	$T_{230} + T_{301} + T_{012} + T_{123}$
	3210	$T_{3210} + T_{3212} + T_{3213} + T_{2101} + T_{2102} + T_{2103} + T_{1030} + T_{1031} + T_{1032} + T_{0320} + T_{0321} + T_{0323}$
	0310	$T_{0310} + T_{0213} + T_{1302} + T_{1021} + T_{2132} + T_{3203}$
	2310	$T_{2310} + T_{3201} + T_{3230} + T_{3231} + T_{0301} + T_{0302} + T_{0312} + T_{0210} + T_{0212} + T_{0230} + T_{0231} + T_{0232} + T_{1301} + T_{1303} + T_{1320} + T_{1321} + T_{1323} + T_{1012} + T_{1013} + T_{1023} + T_{2120} + T_{2123}$
	1230	$T_{1230} + T_{1232} + T_{1231} + T_{2303} + T_{2302} + T_{2301} + T_{3010} + T_{3013} + T_{3012} + T_{0120} + T_{0123} + T_{0121}$

A.3.4 *Table of* $g_\lambda^{(k)}$. We index $g_\lambda^{(k)}$ by k-bounded partitions.

n	λ	$g_\lambda^{(n-1)}$ in terms of s_λ	g_w in terms of $s_\lambda^{(n-1)}$
2	1	s_1	$s_1^{(1)}$
	11	$s_1 + s_{11} + s_2$	$s_1^{(1)} + s_{11}^{(1)}$
	111	$s_1 + 2s_{11} + s_{111} + 2s_2 + s_3 + 2s_{21}$	$s_1^{(1)} + 2s_{11}^{(1)} + s_{111}^{(1)}$
	1111	$s_1 + 3s_{11} + 3s_{111} + s_{1111} + 3s_2 + 3s_3 + 6s_{21} + 3s_{31} + 2s_{22} + 3s_{211} + s_4$	$s_1^{(1)} + 3s_{11}^{(1)} + 3s_{111}^{(1)} + s_{1111}^{(1)}$
	11111	$s_1 + 4s_{11} + 6s_{111} + 4s_{1111} + s_{11111} + 4s_2 + 6s_3 + 12s_{21} + 12s_{31} + 8s_{22} + 12s_{211} + 4s_4 + s_5 + 4s_{41} + 6s_{311} + 5s_{221} + 4s_{2111} + 5s_{32}$	$s_1^{(1)} + 4s_{11}^{(1)} + 6s_{111}^{(1)} + 4s_{1111}^{(1)} + s_{11111}^{(1)}$
3	1	s_1	$s_1^{(2)}$
	2	s_2	$s_2^{(2)}$
	11	$s_1 + s_{11}$	$s_1^{(2)} + s_{11}^{(2)}$
	21	$s_2 + s_{21} + s_3$	$s_2^{(2)} + s_{21}^{(2)}$
	111	$s_1 + s_2 + 2s_{11} + s_{21} + s_{111}$	$s_1^{(2)} + 2s_{11}^{(2)} + s_{111}^{(2)} + s_2^{(2)}$
	22	$s_2 + s_{21} + s_{22} + s_3 + s_4 + s_{31}$	$s_2^{(2)} + s_{21}^{(2)} + s_{22}^{(2)}$
	211	$s_{21} + s_{211} + s_3 + s_{31}$	$s_{21}^{(2)} + s_{211}^{(2)}$
	1111	$s_1 + 2s_2 + 3s_{11} + 3s_{21} + 3s_{111} + s_{22} + s_{211} + s_{1111}$	$s_1^{(2)} + 3s_{11}^{(2)} + 3s_{111}^{(2)} + s_{1111}^{(2)} + 2s_2^{(2)}$
	221	$s_2 + 2s_{21} + 2s_{22} + s_{211} + s_{221} + 2s_3 + 2s_4 + 3s_{31} + s_{311} + 2s_{32} + s_5 + 2s_{41}$	$s_2^{(2)} + 2s_{21}^{(2)} + s_{211}^{(2)} + 2s_{22}^{(2)} + s_{221}^{(2)}$
	2111	$2s_{21} + s_{22} + 3s_{211} + s_{221} + s_{2111} + 2s_3 + s_4 + 4s_{31} + 2s_{311} + s_{32} + s_{41}$	$2s_{21}^{(2)} + 3s_{211}^{(2)} + s_{2111}^{(2)} + s_{22}^{(2)}$
	11111	$s_1 + 3s_2 + 4s_{11} + 8s_{21} + 6s_{111} + 4s_{22} + 7s_{211} + 4s_{1111} + 2s_{221} + 2s_{2111} + s_{11111} + 2s_3 + 3s_{31} + s_{311} + s_{32}$	$s_1^{(2)} + 4s_{11}^{(2)} + 6s_{111}^{(2)} + 4s_{1111}^{(2)} + s_{11111}^{(2)} + 3s_2^{(2)} + 2s_{21}^{(2)} + 3s_{211}^{(2)}$
4	1	s_1	$s_1^{(3)}$
	2	s_2	$s_2^{(3)}$
	11	$s_1 + s_{11}$	$s_1^{(3)} + s_{11}^{(3)}$
	3	s_3	$s_3^{(3)}$
	21	$s_2 + s_{21}$	$s_2^{(3)} + s_{21}^{(3)}$
	111	$s_1 + 2s_{11} + s_{111}$	$s_3^{(3)} + 2s_{11}^{(3)} + s_{111}^{(3)}$
	31	$s_3 + s_{31} + s_4$	$s_3^{(3)} + s_{31}^{(3)}$
	22	$s_2 + s_{21} + s_{22}$	$s_2^{(3)} + s_{21}^{(3)} + s_{22}^{(3)}$
	211	$s_2 + 2s_{21} + s_{211} + s_3 + s_{31}$	$s_2^{(3)} + 2s_{21}^{(3)} + s_{211}^{(3)} + s_3^{(3)}$
	1111	$s_1 + 3s_{11} + 3s_{111} + s_{1111} + s_2 + 2s_{21} + s_{211}$	$s_1^{(3)} + 3s_{11}^{(3)} + 3s_{111}^{(3)} + s_{1111}^{(3)} + s_2^{(3)} + 2s_{21}^{(3)}$

A.3.5 *Table for the coproduct of* $g_\lambda^{(k)}$. The following table gives $\Delta(g_\lambda^{(k)}) = \sum_{\nu,\mu} c_\lambda^{\nu\mu} g_\nu^{(k)} \otimes g_\mu^{(k)}$, where we suppress the superscript '(k)' and write simply $g_\nu \otimes g_\mu$ for $g_\nu^{(k)} \otimes g_\mu^{(k)}$, with ν and μ being k-bounded partitions.

n	λ	$\Delta(g_\lambda^{(n-1)})$
2	1	$g_1 \otimes g_\emptyset + g_\emptyset \otimes g_1$
	11	$g_{11} \otimes g_\emptyset + 2g_1 \otimes g_1 + g_\emptyset \otimes g_{11}$
	111	$g_{111} \otimes g_\emptyset + 3g_{11} \otimes g_1 + 3g_1 \otimes g_{11} + g_\emptyset \otimes g_{111} - 2g_1 \otimes g_1$
	1111	$g_{1111} \otimes g_\emptyset + 4g_{111} \otimes g_1 + 6g_{11} \otimes g_{11} + 4g_1 \otimes g_{111} + g_\emptyset \otimes g_{1111} - 5g_{11} \otimes g_1$ $- 5g_1 \otimes g_{11} + 2g_1 \otimes g_1$
	11111	$g_{11111} \otimes g_\emptyset + 5g_{1111} \otimes g_1 + 10g_{111} \otimes g_{11} + 10g_{11} \otimes g_{111} + 5g_1 \otimes g_{1111}$ $+ g_\emptyset \otimes g_{11111} - 9g_{111} \otimes g_1 - 16g_{11} \otimes g_{11} - 9g_1 \otimes g_{111} + 7g_{11} \otimes g_1$ $+ 7g_1 \otimes g_{11} - 2g_1 \otimes g_1$
3	1	$g_1 \otimes g_\emptyset + g_\emptyset \otimes g_1$
	22	$g_2 \otimes g_\emptyset + g_1 \otimes g_1 + g_\emptyset \otimes g_2$
	11	$g_{11} \otimes g_\emptyset + g_1 \otimes g_1 + g_\emptyset \otimes g_{11}$
	21	$g_{21} \otimes g_\emptyset + g_{11} \otimes g_1 + 2g_2 \otimes g_1 + g_1 \otimes g_{11} + 2g_1 \otimes g_2 + g_\emptyset \otimes g_{21} - g_1 \otimes g_1$
	111	$g_{111} \otimes g_\emptyset + 2g_{11} \otimes g_1 + g_2 \otimes g_1 + 2g_1 \otimes g_{11} + g_\emptyset \otimes g_{111} + g_1 \otimes g_2 - g_1 \otimes g_1$
	22	$g_{22} \otimes g_\emptyset + 2g_{21} \otimes g_1 + g_{11} \otimes g_{11} + g_2 \otimes g_{11} + g_{11} \otimes g_2 + 3g_2 \otimes g_2$ $+ 2g_1 \otimes g_{21} + g_\emptyset \otimes g_{22} - g_2 \otimes g_1 - g_1 \otimes g_2$
	211	$g_{211} \otimes g_\emptyset + g_{111} \otimes g_1 + g_{21} \otimes g_1 + g_{11} \otimes g_{11} + 2g_2 \otimes g_{11} + g_1 \otimes g_{111}$ $+ 2g_{11} \otimes g_2 + g_2 \otimes g_2 + g_1 \otimes g_{21} + g_\emptyset \otimes g_{211} - 2g_{11} \otimes g_1 - 2g_2 \otimes g_1$ $- 2g_1 \otimes g_{11} - 2g_1 \otimes g_2 + g_1 \otimes g_1$
	1111	$g_{1111} \otimes g_\emptyset + 2g_{111} \otimes g_1 + 3g_{11} \otimes g_{11} + g_2 \otimes g_{11} + 2g_1 \otimes g_{111} + g_\emptyset \otimes g_{1111}$ $+ g_{11} \otimes g_2 + g_2 \otimes g_2 - g_{11} \otimes g_1 - g_1 \otimes g_{11}$
4	1	$g_1 \otimes g_\emptyset + g_\emptyset \otimes g_1$
	2	$g_2 \otimes g_\emptyset + g_1 \otimes g_1 + g_\emptyset \otimes g_2$
	11	$g_{11} \otimes g_\emptyset + g_1 \otimes g_1 + g_\emptyset \otimes g_{11}$
	3	$g_3 \otimes g_\emptyset + g_2 \otimes g_1 + g_2 \otimes g_1 + g_\emptyset \otimes g_3$
	21	$g_{21} \otimes g_\emptyset + g_{11} \otimes g_1 + g_2 \otimes g_1 + g_1 \otimes g_{11} + g_1 \otimes g_2 + g_\emptyset \otimes g_{21} - g_1 \otimes g_1$
	111	$g_{111} \otimes g_\emptyset + g_{11} \otimes g_1 + g_1 \otimes g_{11} + g_\emptyset \otimes g_{111}$
	31	$g_{31} \otimes g_\emptyset + g_{21} \otimes g_1 + 2g_3 \otimes g_1 + g_2 \otimes g_{11} + g_{11} \otimes g_2 + 2g_2 \otimes g_2$ $+ g_1 \otimes g_{21} + 2g_1 \otimes g_3 + g_\emptyset \otimes g_{31} - g_2 \otimes g_1 - g_1 \otimes g_2$
	22	$g_{22} \otimes g_\emptyset + g_{21} \otimes g_1 + g_{11} \otimes g_{11} + g_2 \otimes g_2 + g_1 \otimes g_{21} + g_\emptyset \otimes g_{22}$
	211	$g_{211} \otimes g_\emptyset + g_{111} \otimes g_1 + 2g_{21} \otimes g_1 + g_3 \otimes g_1 + g_{11} \otimes g_{11} + 2g_2 \otimes g_{11}$ $+ g_1 \otimes g_{111} + 2g_{11} \otimes g_2 + g_2 \otimes g_2 + 2g_1 \otimes g_{21} + g_\emptyset \otimes g_{211} + g_1 \otimes g_3$ $- g_{11} \otimes g_1 - g_2 \otimes g_1 - g_1 \otimes g_2 - g_1 \otimes g_{11}$
	1111	$g_{1111} \otimes g_\emptyset + 2g_{111} \otimes g_1 + g_{21} \otimes g_1 + 2g_{11} \otimes g_{11} + g_2 \otimes g_{11} + 2g_1 \otimes g_{111}$ $+ g_\emptyset \otimes g_{1111} + g_{11} \otimes g_2 + g_1 \otimes g_{21} - g_{11} \otimes g_1 - g_1 \otimes g_{11}$

A.3.6 *Table of* $G_\lambda^{(k)}$. We have suppressed the superscript '(k)' on $F_\mu^{(k)}$ in the following table.

n	λ	$G_\lambda^{(n-1)}$ in terms of s_λ	$G_\lambda^{(n-1)}$ in terms of $F_\lambda^{(n-1)}$
2	1	$s_1 - s_{1^2} + s_{1^3} - s_{1^4} + s_{1^5} - s_{1^6} \pm \cdots$	$F_1 - F_{1^2} + F_{1^3} - F_{1^4}$ $+ F_{1^5} - F_{1^6} \pm \cdots$
	11	$s_{1^2} - 2s_{1^3} + 3s_{1^4} - 4s_{1^5} + 5s_{1^6}$ $- 6s_{1^7} + \pm \cdots$	$F_{1^2} - 2F_{1^3} + 3F_{1^4} - 4F_{1^5}$ $+ 5F_{1^6} - 6F_{1^7} \pm \cdots$
	111	$s_{1^3} - 3s_{1^4} + 6s_{1^5} - 10s_{1^6} + 15s_{1^7}$ $- 21s_{1^8} \pm \cdots$	$F_{1^3} - 3F_{1^4} + 6F_{1^5} - 10F_{1^6}$ $+ 15F_{1^7} - 21F_{1^8} \pm \cdots$
	1111	$s_{1^4} - 4s_{1^5} + 10s_{1^6} - 20s_{1^7} + 35s_{1^8}$ $- 56s_{1^9} \pm \cdots$	$F_{1^4} - 4F_{1^5} + 10F_{1^6} - 20F_{1^7}$ $+ 35F_{1^8} - 56F_{1^9} \pm \cdots$
	11111	$s_{1^5} - 5s_{1^6} + 15s_{1^7} - 35s_{1^8} + 70s_{1^9}$ $- 126s_{1^{10}} \pm \cdots$	$F_{1^5} - 5F_{1^6} + 15F_{1^7} - 35F_{1^8}$ $+ 70F_{1^9} - 126F_{1^{10}} \pm \cdots$
3	1	$s_1 - s_{1^2} + s_{1^3} - s_{1^4} + s_{1^5} - s_{1^6} \pm \cdots$	$F_1 - F_{1^2} + F_{1^3} - F_{1^4}$ $+ F_{1^5} - F_{1^6} \pm \cdots$
	2	$s_2 - s_{21} + s_{211} - s_{2111} + s_{21111}$ $- s_{211111} \pm \cdots$	$F_2 - F_{21} - F_{111} + F_{211} + F_{1^4}$ $- F_{21^3} - 2F_{1^5} + F_{21^4} + 2F_{1^6} \pm \cdots$
	11	$s_{1^2} - 2s_{1^3} + 3s_{1^4} - 4s_{1^5} + 5s_{1^6} - 6s_{1^7} \pm \cdots$	$F_{1^2} - 2F_{1^3} + 3F_{1^4} - 4F_{1^5}$ $+ 5F_{1^6} - 6F_{1^7} \pm \cdots$
	21	$-s_{1^3} + 2s_{1^4} - 3s_{1^5} + s_{21} - s_{2^2} - s_{21^2} + s_{2^21}$ $+ s_{21^3} + 4s_{1^6} - s_{2^21^2} - s_{21^4} - 5s_{1^7} + s_{21^5}$ $+ s_{2^21^3} - s_{21^6} + 6s_{1^8} - s_{2^21^4} \pm \cdots$	$F_{21} - F_{22} - F_{211} + F_{221}$ $+ 2F_{21^3} + F_{1^5} - F_{2211} - 2F_{21^4}$ $- F_{1^6} + F_{221^3} + 3F_{21^5} + 3F_{1^7} \pm \cdots$
	111	$s_{1^3} - 3s_{1^4} + 6s_{1^5} - 10s_{1^6} + 15s_{1^7}$ $- 21s_{1^8} \pm \cdots$	$F_{1^3} - 3F_{1^4} + 6F_{1^5} - 10F_{1^6}$ $+ 15F_{1^7} - 21F_{1^8} \pm \cdots$
	22	$-s_{1^4} + 2s_{1^5} + s_{2^2} - 2s_{2^21} + s_{21^3} - 3s_{1^6}$ $+ s_{2^3} 2s_{2^2+1^2} - 2s_{21^4} + 4s_{1^7} + 3s_{21^5}$ $- s_{2^31} - 2s_{2^21^3} - 4s_{21^6} - 5s_{1^8} + 2s_{2^21^4}$ $+ s_{2^31^2} - 2s_{2^21^5} + 5s_{21^7} + 6s_{1^9} - s_{2^31^3} \pm \cdots$	$F_{22} - 2F_{221} - F_{21^3} + F_{2^3}$ $+ 2F_{2211} + F_{21^4} - F_{2^31}$ $- 3F_{221^3} - 3F_{21^5} - F_{1^7} \pm \cdots$
	211	$-s_{1^4} + 3s_{1^5} + s_{21^2} - s_{2^21} - 2s_{21^3}$ $- 6s_{1^6} + s_{2^3} + 2s_{2^21^2} + 3s_{21^4} + 10s_{1^7}$ $- 4s_{21^5} - 2s_{2^31} - 3s_{2^21^3} + 5s_{21^6} - 15s_{1^8}$ $+ 4s_{2^21^4} + s_{2^4} + 3s_{2^31^2} - 5s_{2^21^5}$ $- 6s_{21^7} + 21s_{1^9} - 4s_{2^31^3} - 2s_{2^41} \pm \cdots$	$F_{211} - F_{221} - 3F_{21^3} - 3F_{1^5} + F_{2^3}$ $+ 2F_{2211} + 6F_{21^4} + 7F_{1^6} - 2F_{2^31}$ $- 5F_{221^3} - 14F_{21^5} - 25F_{1^7} \pm \cdots$
	1111	$s_{1^4} - 4s_{1^5} + 10s_{1^6} - 20s_{1^7} + 35s_{1^8}$ $- 56s_{1^9} \pm \cdots$	$F_{1^4} - 4F_{1^5} + 10F_{1^6} - 20F_{1^7}$ $+ 35F_{1^8} - 56F_{1^9} \pm \cdots$
	221	$s_{2^21} - s_{21^3} - s_{1^6} - 2s_{2^3} - s_{2^21^2} + 3s_{21^4}$ $+ 3s_{1^7} - 5s_{21^5} + 3s_{2^31} + 7s_{21^6} - 6s_{1^8}$ $+ s_{2^21^4} - s_{2^4} - 3s_{2^31^2} - 2s_{2^21^5} - 9s_{21^7}$ $+ 10s_{1^9} + 3s_{2^31^3} + s_{2^41} - 15s_{1^{10}} - s_{2^41^2}$ $- 3s_{2^31^4} + 3s_{2^21^6} + 11s_{21^8} \pm \cdots$	$F_{221} - F_{2^3} - F_{2211} + 3F_{2^31}$ $+ 3F_{221^3} + F_{21^5} \pm \cdots$

2111	$-2s_{1^5} + s_{21^3} + 7s_{1^6} - s_{2^21^2} - 2s_{21^4} - 16s_{1^7} + 3s_{21^5} + s_{2^31} + 2s_{2^21^3} - 4s_{21^6} + 30s_{1^8} - 3s_{2^21^4} - s_{2^4} - 2s_{2^31^2} + 4s_{2^21^5} + 5s_{21^7} - 50s_{1^9} + 3s_{2^31^3} + 2s_{2^41} + 77s_{1^{10}} - s_{2^5} - 3s_{2^41^2} - 4s_{2^31^4} - 5s_{2^21^6} - 6s_{21^8} \pm \cdots$	$F_{21^3} - F_{2211} - 3F_{21^4} + F_{2^31} + 3F_{221^3} + 9F_{21^5} + 7F_{1^7} \pm \cdots$
11111	$s_{1^5} - 5s_{1^6} + 15s_{1^7} - 35s_{1^8} + 70s_{1^9} - 126s_{1^{10}} \pm \cdots$	$F_{1^5} - 5F_{1^6} + 15F_{1^7} - 35F_{1^8} \pm \cdots$

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