Annals of Mathematics, **148** (1998), 573–617

Level spacings distribution for large random matrices: Gaussian fluctuations*

By Alexander Soshnikov

Abstract

We study the level-spacings distribution for eigenvalues of large $N \times N$ matrices from the classical compact groups in the scaling limit when the mean distance between nearest eigenvalues equals 1.

Defining by $\eta_N(s)$ the number of nearest neighbors spacings greater than $s > 0$ (smaller than $s > 0$) we prove functional limit theorem for the process $(\eta_N(s) - \mathbb{E}\eta_N(s))/N^{1/2}$, giving weak convergence of this distribution to some Gaussian random process on $[0, \infty)$.

The limiting Gaussian random process is universal for all classical compact groups. It is Hölder continuous with any exponent less than $1/2$. Similar results can be obtained for the n-level-spacings distribution.

1. Introduction and formulation of main results

The idea that statistical behavior of eigenvalues of large random matrices would give information about spectra of heavy nuclei was proposed by E. Wigner in the fifties ([37], [38], [39], [40]). Since then, random matrices have been intensively studied by F. J. Dyson, M. L. Mehta, C. E. Porter, N. Rosenzweig, M. Gaudin, L. Pastur, L. Girko and many others. Reference [30] contains an extensive collection of early papers on this subject.

One of the most popular ensembles of random matrices, the so-called Circular Unitary Ensemble (C.U.E.) was investigated by Freeman J. Dyson [10] for studying quantum systems without time reversal symmetry. C.U.E. is the unitary group $\mathrm{U}(N)$ with the normalized translation invariant (Haar) measure. It is a classical result ([35]) that the joint probability distribution of the eigenvalues $\{\exp(i\theta_j)\}_{j=1}^{N}$ in the unitary ensemble is given by the density

$$(1) \qquad P_{N,\beta}(\theta_1, \dots \theta_N) = \mathrm{const}_{N,\beta} \prod_{1 \le k < j \le N} |\exp(i\theta_k) - \exp(i\theta_j)|^{\beta},$$

*AMS Subject classification: Probability theory and stochastic processes.

where the eigenvalues are ordered by increasing their angular coordinates

$$-\pi \leq \theta_1 \leq \ldots \leq \theta_N \leq \pi \tag{2}$$

(here and below we use the segment $[-\pi, \pi]$ with the coinciding ends as the representation for the unitary circle). The Circular Unitary Ensemble corresponds to the case

$$\beta = 2, \qquad \text{const}_{N,2} = (2\pi)^{-N}$$

which is the simplest from the mathematical point of view among all possible choices of β. Two other cases with clear physical meaning, $\beta = 1$ and $\beta = 4$, correspond to the so-called Circular Orthogonal Ensemble (C.O.E.) and the Circular Symplectic Ensemble (C.S.E.) (no relation to the distribution of eigenvalues in the Orthogonal Group $O(N)$ and the Unitary Symplectic Group $USp(2N)$, which will be studied later). It is worth mentioning that from the statistical mechanics point of view one can think about (1) as an equilibrium distribution at the temperature $T = 1/\beta$ of the Coulomb gas of N unit charges, confined to the infinitely thin circular conducting wire of radius 1, repelling each other according to the Coulomb law of two-dimensional electrostatics, i.e. with potential energy

$$W = -\sum_{1 \leq k < j \leq N} \log |\exp(i\theta_k) - \exp(i\theta_j)|.$$

Due to the logarithmic repulsion, typical configurations of the particles are very regularly distributed on the unit circle. For example, if we consider the number of particles hitting the interval

$$(-x, x) \subset [-\pi, \pi] \quad , \quad \mu_N(x) = \#\{j : |\theta_j| < x\},$$

then the mathematical expectation of $\mu_N(x)$ is proportional to the number N of all particles, $\mathbb{E}\mu_N(x) = Nx/\pi$, but the variance Var $\mu_N(x)$ grows only logarithmically,

$$\text{Var } \mu_N(x) = \frac{2 \log N}{\pi^2 \beta} + O(1), \qquad \beta = 1, 2, 4.$$

After the normalization, the random variable

$$\Big(\mu_N(x) - \mathbb{E}\mu_N(x)\Big) / \Big(\text{Var } \mu_N(x)\Big)^{1/2}$$

converges to the standard Gaussian random variable. This and similar results can be found in the papers by O. Costin and J. Lebowitz [8], K. Johansson [16], [17], [18], H. Spohn [33], P. Diaconis and M. Shahshahani [9], T. H. Baker and P. J. Forrester [1], E. Basor [2].

With the exception of [16], [18], [1] the results have been obtained so far only for $\beta = 1, 2, 4$. Some heuristic arguments for the case of general β have been devised in [12], [14]. The main goal of our paper is to study the statistical behavior of level spacings for the Circular Unitary Ensemble ($\beta = 2$).[1] After ordering in (2) the eigenvalues by increasing their angular coordinates, we can define the nearest neighbor spacings as

$$\tau_j = \theta_{j+1} - \theta_j, \quad j = 1, \ldots, N-1; \qquad \tau_N = \theta_1 + 2\pi - \theta_N. \tag{3}$$

The n-point correlation functions ($n = 1, \ldots N$) of our ensemble

$$\rho_n^{(N)}(x_1, \ldots x_n) = \frac{1}{(N-n)!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} P_{N,2}(x_1, \ldots x_N) dx_{n+1} \ldots dx_N$$

(we extend the domain of definition of $P_{N,2}$ by symmetry to the whole N-dimensional torus) have the following probabilistic meaning: let $[x_1, x_1 + dx_1], \ldots [x_n, x_n + dx_n]$ be n-infinitesimally small disjoint intervals, then $\rho_n^{(N)}(x_1, \ldots x_n) dx_1 \ldots dx_n$ is the probability to find eigenvalues in each of them.

For $\beta = 1, 2, 4$, n-point correlation functions have been calculated explicitly by F. Dyson (see [10], [11]) and in the case of C.U.E.

$$\rho_n^{(N)}(x_1, \ldots x_n) = (1/2\pi)^n \det \left(\frac{\sin N(x_i - x_j)/2}{\sin (x_i - x_j)/2} \right)_{i,j=1,\ldots n} \tag{4}$$

The conditional probability of having no eigenvalues in the interval $(0, u]$ provided there is an eigenvalue at the origin (that is, the probability of nearest-neighbor spacing τ to be greater than u) can be calculated by the inclusion-exclusion principle:

(5)
$$\begin{aligned}\mathbb{P}_N(\tau > u) = \Big(\rho_1^{(N)}(0) &- \int_0^u \rho_2^{(N)}(0, x_2) dx_2 + \frac{1}{2!} \int_0^u \int_0^u \rho_3^{(N)}(0, x_2, x_3) dx_2 dx_3 \\ &- \frac{1}{3!} \int_0^u \int_0^u \int_0^u \rho_4^{(N)}(0, x_2, x_3, x_4) dx_2 dx_3 dx_4 + \cdots \Big) / \rho_1^{(N)}(0).\end{aligned}$$

The mean distance between the nearest eigenvalues in C.U.E. is equal to $2\pi N$. After a suitable rescaling (extension by $N(2\pi)$ times the segment $[-\pi, \pi]$), this distance becomes equal to 1. In the new coordinates

$$y_k = N/2 + N\theta_k/(2\pi), \qquad k = 1, \ldots N,$$

the rescaled n-point correlation functions

$$(2\pi/N)^n \rho_n^{(N)}(2\pi y_1/N, \ldots 2\pi y_n/N) = \det \left(\frac{\sin \pi(y_i - y_j)}{N \sin(\pi(y_i - y_j)/N)} \right)_{i,j=1,\ldots n} \tag{6}$$

[1] We learned from N. Katz and P. Sarnak [19] that our methods can also be applied to study other classical compact groups:

$$\mathrm{SO}(2N), \mathrm{SO}(2N+1), \mathrm{O}(2N), \mathrm{O}(2N+1), \mathrm{USp}(2N), \mathrm{SU}(N), \mathrm{O}_-(2N)$$

(see §5 for the corresponding results). Similar results for the Circular Orthogonal Ensemble ($\beta = 1$) are discussed in Section 6.

have a finite limit as N tends to infinity:

(7)
$$\lim_{N\to\infty}(2\pi/N)^n\rho_n^{(N)}(2\pi y_1/N,\dots 2\pi y_n/N) =: \rho_n^{(\infty)}(y_1,\dots y_n) = \det\left(\frac{\sin\pi(y_i-y_j)}{\pi(y_i-y_j)}\right)_{i,j=1,\dots n}$$

and respectively for

$$F_N(s) := \mathbb{P}_N(\tau > 2\pi s/N)$$

(8)
$$\lim_{N\to\infty} F_N(s) =: F(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\rho_{n+1}^{(\infty)}(0,y_1,\dots y_n)dy_1\dots dy_n.$$

Remark. The limiting correlation functions (7) define a random-point field on the real line, i.e. the probability measure on the Borel σ-algebra of the space of locally finite point configurations

$$\Omega = \left\{(x_i)_{i=-\infty,\dots+\infty} \;:\; \forall L>0 \;\; \#\{x_i : |x_i| < L\} < \infty\right\}$$

in the following way: If we fix m disjoint intervals $[a_{2j-1}, a_{2j}]_{j=1,\dots m}$ and define random variables $\mu_1,\dots \mu_m$ to be the numbers of particles hitting each interval, then the generating function

$$\varphi(z_1,\dots z_m) := \mathbb{E}\prod_{j=1}^{m} z_j^{\mu_j}$$

is given by the Fredholm determinant of the integral operator acting on $L^2(R^1)$, with the kernel

$$\sum_{i=1}^{m}(z_j-1)\frac{\sin\pi(x-y)}{\pi(x-y)}\mathbf{J}_j(y)$$

where the $\mathbf{J}_j$ are indicators of the segments $[a_{2j-1}, a_{2j}]$ (see [34]). Such a defined random-point field is called a Universal Random Matrix Limit (URML) in the literature of physics. It was conjectured by Dyson to be the limiting case for the general unitary invariant ensembles of hermitian matrices (see [29], [4], [7] for recent results).

Remark. Function $F(s)$ decays at infinity superexponentially:

$$\log F(s) = -\pi^2 s^2/8 + O(s)$$

(see [24, Chap. 12], also [34], [36]).

Recently, Z. Rudnick and P. Sarnak ([31]) showed that after a suitable rescaling the n-point correlation functions for zeroes of the Riemann zeta function on the critical line $\Re e\, z = 1/2$ are given exactly by the same formula (7). These results are valid in a restricted range; see also the early paper on pair-correlations by H. L. Montgomery ([26]), and numerical results by A. M. Odlyzko on the spacings distribution of zeroes ([27], [28]). We finish this section with the formulation of our main results:

THEOREM 1.1. *Consider an arbitrary subinterval I_N of the unit circle such that the average number of eigenvalues hitting the subinterval tends to infinity as $N \to \infty$: $|I_N|N/(2\pi) \to \infty$. Define $\eta(I_N, s)$ to be the number of eigenvalues belonging to I_N for which the distance to the nearest right neighbor is greater (smaller) than $2\pi\, s/N$:*

$$\eta(I_N, s) := \#\{j : \theta_j \in I_N, \quad \tau_j = \theta_{j+1} - \theta_j > (<) 2\pi s/N\}.$$

Then

$$\mathbb{E}\eta(I_N, s) = \frac{N|I_N|}{2\pi}\mathbb{P}_N\Big(\tau > \ (<) 2\pi s/N\Big)$$

and finite-dimensional distributions of the normalized random process

$$\xi_N(s) = \Big(\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)\Big)/\ (N|I_N|/2\pi)^{1/2}$$

converge to the distributions of the Gaussian random process with $\mathbb{E}\xi(s) = 0$ and $b(s,t) := \mathbb{E}\xi(s)\xi(t)$ given by the formulas (37), (38), (26) *in Section* 3.

To formulate the results about functional convergence we have to define the continuous approximation of $\xi_N(s)$. The realizations of $\eta(I_N, s)$ have discontinuities at points

$$\frac{N}{2\pi}\tau_j = \frac{N}{2\pi}(\theta_{j+1} - \theta_j), \quad \theta_j \in I_N : \quad \eta(I_N, \frac{N}{2\pi}\tau_j + 0) - \eta(I_N, \frac{N}{2\pi}\tau_j) = -1.$$

We define the graph of $\tilde{\eta}(I_N, s)$ by linearly connecting the neighboring vertices $(\frac{N}{2\pi}\tau_j, \eta(I_N, \frac{N}{2\pi}\tau_j)), \theta_j \in I_N$, and

$$\tilde{\xi}_N(s) := \Big(\tilde{\eta}(I_N, s) - \mathbb{E}\eta(I_N, s)\Big)/\Big(N|I_N|/(2\pi)\Big)^{1/2}.$$

The distribution of $\tilde{\xi}_N(\cdot)$ defines a probability measure $\mathcal{P}_N$ on the space of continuous functions $C[0,\infty)$ (infinity point is not included!).

THEOREM 1.2. *$\mathcal{P}_N$ weakly converges to the distribution of the Gaussian process $\xi(\cdot)$.*

Of course in both theorems we can take I_N to be $[-\pi, \pi]$. In this case $\eta_N([-\pi, \pi], s)$ will count all nearest-neighbor spacings greater(smaller) than s.

COROLLARY 1.3. *If we consider disjoint intervals*

$$I_N^{(1)}, \ldots, I_N^{(m)}$$

such that

$$0 < \text{const}_1 \le |I_N^{(i)}|/\ |I_N^{(j)}| \le \text{const}_2 < \infty, \quad i,j = 1, \ldots m$$

and

$$N|I_N^{(i)}|/(2\pi) \to \infty \qquad as \quad N \to \infty$$

then a random vector

$$\left(\left(\eta(I_N^{(i)}, s_i) - \mathbb{E}\eta(I_N^{(i)}, s_i)\right)/(\mathrm{Var}\ \eta(I_N^{(i)}, s_i))^{1/2}\right)_{i=1}^{m}$$

converges in distribution to the standard Gaussian random vector with independent components.

COROLLARY 1.4. *For any finite* $T > 0$,

$$\sup_{s\in[0,T]} \quad |\eta(I_N, s) - \mathbb{E}\eta(I_N, s)|/(N|I_N|/(2\pi))^{1/2} \tag{9}$$

converges in distribution to

$$\sup_{s\in[0,T]} |\xi(s)|.$$

Remark. Since

$$\sup_{[0,\infty)} \left|\mathbb{E}\eta(I_N, s) - N|I_N|\ F(s)/(2\pi)\right| = o\Big(|I_N|N^{\varepsilon}/(2\pi)\Big)$$

for any $\varepsilon > 0$ (see Lemma 4.2), one can replace in (9) $\mathbb{E}\eta(I_N, s)$ by $F(s)N|I_N|/(2\pi)$.

We have not been able to prove the result of Corollary 1.4 for $T = \infty$ (the functional convergence of probability distributions is proven for $C[0,\infty)$, not for $C[0,\infty]$!). Therefore we settle for a weaker version:

COROLLARY 1.5. *With probability* 1,

$$\sup_{[0,\infty)} \left|\eta(I_N, s) - \mathbb{E}\eta(I_N, s)\right| = o\left(\left(N|I_N|/(2\pi)\right)^{1/2\ +\varepsilon}\right) \tag{10}$$

for any $\varepsilon > 0$. *The same estimate also holds for the mathematical expectation of the left-hand side in* (10).

Remark. The discrepancy at the left-hand side of (10) was studied for the first time by N. Katz and P. Sarnak who did it in connection with the theory of geometric zeta functions over finite fields (see [19]). They proved the estimate

$$\mathbb{E}\ \sup_{[0,\infty)} \left|\eta(I_N, s) - \mathbb{E}\eta(I_N, s)\right| = o\left(\left(N|I_N|/(2\pi)\right)^{5/6\ +\varepsilon}\right)$$

to show that for typical geometric zeta functions the empirical distribution functions of the normalized spacings converge to the Gaudin law $F(s)$.

Remark. Again we can replace $\mathbb{E}\eta(I_N, s)$ by $F(s)N|I_N|/(2\pi)$. As usual for the C.U.E., similar results also hold for the limiting random-point field (7):

THEOREM 1.6. *Consider the number of particles hitting the interval* $[0, L]$ *for which the distance to the nearest right neighbor is greater than* s:

$$\eta(L, s) = \#\Big\{x_i : 0 \le x_i \le L\ , \mathrm{dist}(x_i, \text{right ngb }(x_i)) > s\Big\}.$$

Then $\mathbb{E}\eta(L, s) = LF(s)$ *and*

$$\xi_L(s) = \Big(\eta(L, s) - LF(s)\Big)/L^{1/2}$$

converges in finite-dimensional distributions to the Gaussian random process of Theorem 1.1.

Again we can define piecewise linear continuous approximation $\tilde{\xi}_L(s)$ of $\xi_L(s)$ such that

$$|\tilde{\xi}_L(s) - \xi_L(s)| \le L^{-1/2}$$

and, as the analogue of Theorem 1.2, we have:

THEOREM 1.7. *The distribution of* $\tilde{\xi}_L(\cdot)$ *on* $C[0, \infty)$ *weakly converges to the distribution of* $\xi(\cdot)$.

Remark. We do not know simple "explicit" formulas for the covariance function $\mathbb{E}\xi(s)\xi(t)$ of the limiting Gaussian process. Since

$$\mathbb{E}(\xi(t + \delta t) - \xi(t))^2 = O(|\delta t|)$$

uniformly on any finite interval $t \in [0, T]$, $\xi(s)$ is Hölder continuous with any exponent $\alpha < 1/2$. The numerical results by S. Miller ([25]) suggest that $\xi(s)$ is not a standard Brownian bridge, which would be the case had the spacings been independent random variables.

The proofs of Theorems 1.1 and 1.6, Theorems 1.2 and 1.7 are almost identical. In the next section we will discuss all necessary prerequisites concerning n-point correlation and Ursell functions. We will prove Theorem 1.1 in Section 3. The proofs of Theorem 1.2 and corollaries are given in Section 4. Results similar to Theorems 1.1 and 1.2 are valid in the case $\beta = 1$ and for orthogonal and symplectic groups. Minor changes, required in the formulations and proofs of the theorems are discussed in Sections 5 and 6. Section 7 is devoted to generalizations and concluding remarks. I would like to express my sincere gratitude to my advisor Ya. Sinai and to M. Aizenman, P. Sarnak and H. Spohn for many useful discussions. I would also like to thank N. Katz and P. Sarnak for providing me with their notes on the subject ([19]) prior to publication.

2. Random-point fields on the real line. Correlations and Ursell functions

In this section we give an exposition of some basic facts about random-point fields on the real line (for a more detailed account see [21], [22], [20], [8]). We consider the space of locally finite configurations

$$\Omega = \left\{ \omega = (x_i)_{i=-\infty,\dots+\infty} \ : \ \forall L > 0 \ \ \#\{x_i : |x_i| < L\} < \infty \right\},$$

and reserve the notation η_A for the number of particles in $A \subset R^1$.

The class of measurable sets in Ω is defined as the minimal σ-algebra containing all $\{\omega : \eta_A(\omega) = k\}$, where k is a nonnegative integer and A is a measurable subset of the real line.

Assume a probability measure on Ω. If there exists the joint density $\rho_n(x_1, \dots x_n)$ of n-tuples (i.e. $\rho_n(x_1, \dots x_n)\ dx_1 \cdots dx_n$ is the probability of finding a particle in each of the infinitesimally small intervals $[x_1, x_1 + dx_1], \dots$ $[x_n, x_n + dx_n]$) we call ρ_n an n-point correlation function.

It was first pointed by Ruelle ([32]) that in general the sequence of correlation functions $\rho_n, n = 1, 2, \dots$, does not uniquely characterize the underlying probability measure. The existence and uniqueness problems were studied in detail by A. Lenard in [21], [22]. In particular, the criterion for uniqueness is satisfied when $0 \le \rho_n(x_1, \dots x_n) \le c^n n^{2n}$.

An interesting class of correlation functions (see [33]) can be constructed with the help of a nonnegative integrable function

$$v : R^1 \to R^1, \ \ 0 \le v \le 1,$$

if we define

$$\rho_n(x_1, \dots x_n) = \det\left(\hat{v}(x_i - x_j)\right)_{i,j=1,\dots n} \tag{11}$$

where $\hat{v}$ is the Fourier transform of v:

$$\hat{v}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ixk)\ v(k) dk.$$

Choosing v to be the indicator of the segment:

$$v(k) = \chi_{[-\pi,\pi]}(k)$$

we arrive at URML (7):

$$\rho_n(x_1, \dots x_n) = \det\left(\frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}\right)_{i,j=1,\dots n}.$$

If we want to study the number of points (particles) in the interval of length L,

$$\eta(L) = \#\{x_i : x_i \in [0, L]\},$$

it is very helpful to introduce the so-called Ursell functions (see [20], [8])

$$\begin{aligned} r_1(x_1) &= \rho_1(x_1) \\ r_2(x_1,x_2) &= \rho_2(x_1,x_2) - \rho_1(x_1)\,\rho_1(x_2) \\ r_3(x_1,x_2,x_3) &= \rho_3(x_1,x_2,x_3) - \rho_2(x_1,x_2)\,\rho_1(x_3) - \rho_2(x_1,x_3)\,\rho_1(x_2) \\ &\quad - \rho_2(x_2,x_3)\,\rho_1(x_1) + 2\rho_1(x_1)\,\rho_1(x_2)\,\rho_1(x_3). \end{aligned} \tag{12}$$

In general:

$$r_n(x_1, \ldots x_n) = \sum_G (-1)^{m-1}(m-1)! \prod_{j=1}^{m} \rho_{G_j}(\bar{x}(G_j)) \tag{13}$$

where G is a partition of indices $\{1, 2, \ldots n\}$ into m subgroups $G_1, \ldots G_m, m = 1, \ldots n$, and $\bar{x}(G_j)$ are x_i with indices in G_j. Correlation functions can be obtained from the Ursell functions by the inversion formula

$$\rho_n(x_1, \ldots x_n) = \sum_G \prod_{j=1}^{m} r_{G_j}(\bar{x}(G_j)). \tag{14}$$

If we restrict the summation in (14) to the partitions of $\{1, 2, \ldots n\}$ into two or more point subsets, we will get centralized n-point correlation functions. In random matrix literature $(-1)^{k-1} r_k$ are usually called cluster functions (see [10], [11], [24]). In the particular case of URML

(15)

$$r_n(x_1, \ldots x_n) = (-1)^{n-1} \sum_\sigma \frac{\sin \pi(x_2 - x_1)}{\pi(x_2 - x_1)} \frac{\sin \pi(x_3 - x_2)}{\pi(x_3 - x_2)} \cdots \frac{\sin \pi(x_1 - x_n)}{\pi(x_1 - x_n)}$$

where the sum is over all cyclic permutations.

Ursell functions possess a fundamental property of vanishing when variables $x_1, \ldots x_n$ can be decomposed into two nonempty subsets, belonging to the intervals with independent probability distributions. As was pointed out in [20] "all correlations (which are) due to subsets have been subtracted in forming $r_n(x_1, \ldots x_n)$ from $\rho_n(x_1, \ldots x_n)$, leaving only "intrinsic" n-body correlations."

Ursell functions are closely related to the cumulants $C_j(L)$ of the random variable $\eta(L)$: the integral of $r_k(x_1, \ldots x_k)$ over the k-dimensional cube

$$[0, L] \times \ldots \times [0, L] = [0, L]^k$$

is equal to the linear combination of $C_j(L), j = 1, \ldots k$:

$$U_1 = \int_0^L r_1(x)dx = \mathbb{E}\,\eta(L) = C_1(L),$$

$$
\begin{aligned}
U_2 &= \int_0^L \int_0^L r_2(x_1, x_2)\, dx_1 dx_2 \;= \mathbb{E}\, \eta(\eta-1) - (\mathbb{E}\eta)^2 \;= C_2(L) - C_1(L), \\
U_3 &= \int_0^L \int_0^L \int_0^L r_3(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \\
&= \mathbb{E}\, \eta(\eta-1)(\eta-2) - 3\mathbb{E}\, \eta(\eta-1)\, \mathbb{E}\eta + 2(\mathbb{E}\eta)^3 \\
&= C_3(L) - 3C_2(L) + 2C_1(L).
\end{aligned}
$$

To derive the general formula we can use (12), (13) to write the identities for the generating functions

$$
\sum_1^\infty \frac{1}{k!}\, U_k z^k \;=\; \log\Big(1 + \sum_{k=1}^\infty \frac{1}{k!} \mathbb{E}\eta \cdots (\eta - k + 1)\; z^k\Big) \tag{16}
$$

and

$$
\begin{aligned}
\sum_1^\infty \frac{C_k}{k!}\, z^k \;&=\; \log\, \mathbb{E}\, \exp(z\eta) \\
&=\; \log\, \Big(1 + \sum_1^\infty \frac{1}{k!} \mathbb{E}\eta \cdots (\eta - k + 1)\; (e^z - 1)^k\Big) \\
&=\; \sum_1^\infty \frac{1}{k!}\, U_k\, (e^z - 1)^k.
\end{aligned} \tag{17}
$$

Formulas (16)–(17) yield (see ([8]):

$$
C_k = \Big(\sum_{j=1}^{k-1} b_{k,j}\, C_j\Big) + \, U_k \tag{18}
$$

where

$$
\begin{cases}
b_{k,j} = b_{k-1,j-1} - (k-1) b_{k-1,j} & ,\; 2 \le j \le k-1, \\
b_{k,k} = -1 & ,\; k > 2, \\
b_{k,1} = (-1)^k (k-1)!\, .
\end{cases} \tag{19}
$$

As an immediate consequence of (18), (19), the following central limit theorem holds for the number of particles in the box $[0, L], L \to \infty$:

THEOREM 2.1. *Assume the mathematical expectation and the variance of the number of particles in the interval $[0, L]$ are proportional to L as $L \to \infty$, and suppose the integrals $U_k, k > 2$, of the Ursell functions over $[0, L]^k$ do not grow faster than $o(L^{k/2})$. Then the normalized number of particles*

$$
\frac{\eta(L) - \mathbb{E}\eta(L)}{\Big(\mathrm{Var}\; \eta(L)\Big)^{1/2}}
$$

converges in distribution to the Gaussian normal random variable as $L \to \infty$.

Remark. We have not seen this theorem explicitly stated in the mathematical literature before. However, all its necessary ingredients could be found in [8].

Remark. One can see that Theorem 2.1 is not applicable to the case of URML (7), since the variance grows only logarithmically,

$$\mathrm{Var}\ \eta(L) = \left(\frac{1}{\pi^2}\right) \log\ L + \ O(1)$$

(the fact that distinguishes URML from other random-point fields with the determinantal correlation functions (11)). Because of this, Costin and Lebowitz had to use in [8] more subtle arguments to prove the Gaussian fluctuations. Namely they show that

$$\int_0^L \cdots \int_0^L \frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)}\ \frac{\sin\pi(x_2-x_3)}{\pi(x_2-x_3)} \cdots \frac{\sin\pi(x_k-x_1)}{\pi(x_k-x_1)}$$
$$= \ L + o\Big((\log L)^{k/2}\Big), \quad k > 2;$$

that combined with (15), (18), (19) implies $C_k(L) = o\Big((\log L)^{k/2}\Big), k > 2$, and thus finishes the proof.

We will use the general framework of this section, in particular Theorem 2.1 in our analysis of nearest spacings distribution. Let us fix some $s > 0$. To study the number of spacings greater than s in the interval $[0, L]$ we construct an "s-modified random field", keeping only the particles, for which the distance to the nearest right neighbor is greater than s. Now the number of spacings greater than s in the interval $[0, L]$ for the original random-point field is equal to the number of all particles in $[0, L]$ for the modified one. To apply Theorem 2.1 we need to calculate the correlation and Ursell functions of the modified random-point field. This plan is carried out in Section 3, with the conditions of Theorem 2.1 checked in (33), (38) and in Lemma 3.2. In particular we prove that the Ursell functions $r_l(x_1, \dots x_l, s)$ of the s-modified random field allow the estimates

$$|r_l(x_1, \dots x_l, s)|$$
$$\leq \mathrm{const}(s,\varepsilon) \sum_\sigma \left(\frac{1}{|x_2-x_1|+1} \cdot \frac{1}{|x_3-x_2|+1} \cdots \frac{1}{|x_1-x_l|+1}\right)^{1-\varepsilon}$$

which are valid for all $\varepsilon > 0$ and $x_1, \dots x_l$, such that $\min_{i\neq j} |x_i - x_j| > s$. We do not derive estimates on the Ursell functions in the region $\min_{i\neq j} |x_i - x_j| \leq s$, since the combinatorics turns out to be more involved. Rather than that, we do this part of the proof in a more straightforward way by calculating the main term in the centralized correlation functions.

3. Proof of Theorem 1.1

We shall prove Theorem 1.1 by computing all (to be more precise, first N) moments of random variable $\eta(I_N, s) - \mathbb{E}\eta(I_N, s)$. Without loss of generality we may assume the interval I_N to be the unit circle. In the rescaled coordinates

$$\{y_i = N\theta_i/(2\pi) + N/2\}_{i=1}^{N}$$

the N-dimensional probability density (1) and N-point correlation functions are given by the formulas (20) and (21):

$$\begin{aligned} (20)\quad P_{N,2}(y_1, \dots y_N) &= N^{-N} \prod_{1 \le k < j \le N} |\exp(i\pi y_j/N) - \exp(i\pi y_k/N)|^2 \\ &= \det\left(\frac{\sin\pi(y_i - y_j)}{N\sin(\pi(y_i - y_j)/N)}\right)_{i,j=1,\dots N}, \\ & \qquad 0 \le y_1 \le \dots \le y_N \le N \end{aligned}$$

and

$$(21) \qquad \rho_n^{(N)}(y_1, \dots y_n) = \det\left(\frac{\sin\pi(y_i - y_j)}{N\sin(\pi(y_i - y_j)/N)}\right)_{i,j=1,\dots n}.$$

We will omit the index N in the notation for n-point correlation functions if it does not lead to ambiguity; we also consider all variables y_i modulo N. The main aim of this section is to show that

$$\begin{aligned} \mathbb{E}\left(\left(\eta_N(s) - \mathbb{E}\eta_N(s)\right)/N^{1/2}\right)^{2k} &= (2k-1)!!\ (b(s,s))^k + o(1), \\ \mathbb{E}\left(\left(\eta_N(s) - \mathbb{E}\eta_N(s)\right)/N^{1/2}\right)^{2k+1} &= o(1), \end{aligned}$$

where $b(s,s)$ is the variance of the limiting Gaussian process $\xi(s)$.

To calculate the moments of

$$\eta_N(s) := \eta([-\pi, \pi], s)$$

we introduce a representation of $\eta_N(s)$ as "a sum of infinitesimally small random variables." This representation will be used throughout the whole proof. Consider the interval $[0, N]$ as the disjoint union of infinitesimally small subintervals $[x_i, x_i + dx_i]$:

$$[0, N] = \bigcup_i [x_i, x_i + dx_i], \quad x_{i+1} = x_i + dx_i$$

and for each subinterval denote by $\chi(x_i, dx_i, s)$ the indicator of the event to have an eigenvalue in $[x_i, x_i + dx_i]$ and no eigenvalues in $[x_i + dx_i, x_i + s]$. Then

$$(22) \qquad \eta_N(s) = \int_0^N \chi(x, dx, s).$$

More rigorously, (22) means that $\eta_N(s)$ is the integral of the discrete measure $\chi(dx)$ which has unit atoms at the points y_i, such that $y_{i+1} - y_i > s$ (or we can say that $\eta_N(s)$ is the number of points of the s-modified random-point field). The representation of $\eta_N(s)$ as "the sum of weakly dependent random variables"[2] gives us a natural setting for the central limit theorem.

Using the inclusion-exclusion principle, one can calculate the mathematical expectation of the products of $\chi(x_i, dx_i, s), i = 1, \ldots m$. First consider the mathematical expectation of the single term:

$$(23)\quad \mathbb{E}\chi(x_1, dx_1, s) = \Big(\rho_1^{(N)}(x_1) - \int_{x_1}^{x_1+s} \rho_2^{(N)}(x_1, x_2)dx_2 + \frac{1}{2!}\int_{x_1}^{x_1+s}\int_{x_1}^{x_1+s} \rho_3^{(N)}(x_1, x_2, x_3)dx_2dx_3 - \ldots\Big)dx_1 = F_N(s)dx_1.$$

To calculate $\mathbb{E}\chi(x_1, dx_1, s)\chi(x_2, dx_2, s)$ we have to consider two cases: $|x_1 - x_2|_1 \leq s$ and $|x_1 - x_2|_1 > s$. In the former, the mathematical expectation of the product is zero, by definition of $\chi(x, dx, s)$, and in the latter

$$(24)\quad \mathbb{E}\chi(x_1, dx_1, s)\chi(x_2, dx_2, s) = \Big(\sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int \rho_{2+m}(x_1, x_2; \ldots x_{m+2})dx_3 \ldots dx_{m+2}\Big)dx_1dx_2$$

where each variable $x_3, \ldots x_{m+2}$ is integrated over the union of two intervals $[x_1, x_1 + s]$ and $[x_2, x_2 + s]$.

The key combinatorial observation used in the proof can be first seen when we calculate the covariance of $\chi(x_1, dx_1, s), \chi(x_2, dx_2, s)$. We are going to use the cluster structure of n-point correlation functions (21). Consider the m^{th} term in (24) and fix the variables of integration $x_3, \ldots x_{m+2}$.

Some of the x_i, $i = 1, \ldots m+2$, say k of them, $1 \leq k < m+2$, belong to the interval $[x_1, x_1 + s]$; we will denote the indices of those variables by

$$i_1, \ldots i_k; \qquad 1 = i_1 < \ldots < i_k \leq m+2.$$

We will denote the indices of the remaining variables by

$$j_1, \ldots j_{m+2-k}; \qquad 2 = j_1 < \ldots < j_{m+2-k} \leq m+2.$$

It is clear that

$$x_{j_l} \in [x_2, x_2 + s], \qquad l = 1, \ldots m+2-k.$$

[2]We will be able to show that $\mathrm{Cov}(\chi(x_1, dx_1, s), \chi(x_2, dx_2, s)) = g_N(s, |x_1 - x_2|_1)dx_1dx_2$ where $|g_N(s, x)| \leq \mathrm{const}(s, \varepsilon)/(1 + |x|^{2-\varepsilon})$ for any $\varepsilon > 0$ and $|x_1 - x_2|_1 := \min(|x_1 - x_2|, N - |x_1 - x_2|)$.

From (21) it follows that

$$\rho^{(N)}_{2+m}(\bar{x}) = \sum_{\sigma\in S_{m+2}} (-1)^{\sigma} \prod_{i=1}^{m+2} \frac{\sin\pi(x_i - x_{\sigma(i)})}{N\sin(\pi(x_i - x_{\sigma(i)})/N)} \tag{25}$$

(here we use the notation $\bar{x}$ for the vector $(x_1, \dots x_{m+2})$ and σ for permutations from the symmetric group S_{m+2}). Now we decompose sum (25) into two, where the first corresponds to the "interaction" between the particles x_1 and x_2 and is the sum over such $\sigma \in S_{m+2}$, so that

$$\sigma(\{i_1, \dots i_k\}) \cap \{j_1, \dots j_{m+2-k}\} \neq \emptyset,$$

and the second is over all other σ. Denoting the first sum by $\rho_{2+m,2}$ we have

$$\begin{aligned}(26)\ \rho_{2+m}(x_1, x_2, \dots x_{2+m}) &= \rho_{2+m,2}(x_1, x_2, \dots x_{2+m}) \\ &+ \rho_k(x_{i_1}, \dots x_{i_k}) \cdot \rho_{2+m-k}(x_{j_1}, \dots x_{j_{2+m-k}}).\end{aligned}$$

Formulas (23), (24), (26) imply for $|x_1 - x_2|_1 > s$:

$$\begin{aligned}(27)\ &\mathrm{Cov}(\chi(x_1, dx_1, s)\chi(x_2, dx_2, s)) \\ &= \left(\sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int \rho_{2+m,2}(x_1, x_2; \dots x_{m+2}) dx_3 \dots dx_{m+2} + \mathcal{R}\right) dx_1 dx_2\end{aligned}$$

where the remainder term $\mathcal{R}$ would vanish if the summation in (23), (24), (26) were from zero to infinity, and in our case

$$\begin{aligned}(28)\qquad |\mathcal{R}| \leq \sum_{0\leq k_1,k_2\leq N,\ k_1+k_2>N} &\frac{1}{k_1!}\frac{1}{k_2!} \int \rho_{k_1}(x_{i_1}, \dots x_{i_{k_1}}) dx_{i_2} \dots dx_{i_{k_1}} \\ &\int \rho_{k_2}(x_{j_1}, \dots x_{j_{k_2}}) dx_{j_2} \dots dx_{j_{k_2}}.\end{aligned}$$

In (27) the variables $x_3, \dots x_{2+m}$ are integrated over $[x_1, x_1 + s] \cup [x_2, x_2 + s]$; in (28) the variables $x_{i_2}, \dots x_{i_k}$ are integrated over $[x_1, x_1 + s]$, and the variables $x_{j_2}, \dots x_{j_{m+2-k}}$ are integrated over $[x_2, x_2 + s]$.

Remark. Formulas (23), (24) give us one- and two-point correlation functions of the s-modified random-point field. General formulas for the $2k$-point correlation functions are given in (42) and for the centralized $2k$-point correlation functions in (46), Prop. 3.1. Since

$$0 \leq \det\left(\frac{\sin\pi(x_i - x_j)}{N\sin(\pi(x_i - x_j)/N)}\right)_{i,j=1,\dots n} \leq 1, \tag{29}$$

s is bounded throughout the proof of Theorems 1.1, 1.2, and since these arguments only for $s < (\log N)^{1/2}$ in the proof of Corollary 1.5, we obtain the estimate:

$$|\mathcal{R}| \leq N^2(2s)^N 2^N/N! \leq \left(\frac{\mathrm{const}\log N}{N}\right)^N N^2.$$

This inequality shows that we can neglect $\mathcal{R}$ throughout the proof. The upper bound for the determinant in (29) is a general property of n-dimensional positive defined matrices with the trace less than or equal to n. It follows from (26), (29) that

$$(30)\qquad |\rho_{2+m,2}(x_1, x_2, \dots x_{2+m})| \leq 2$$

and the sums (23), (24), (27) are uniformly convergent as $N \to \infty$.

To calculate the variance of $\eta_N(s)$ we need to know how fast

$$\mathrm{Cov}(\chi(x_1, dx_1, s), \chi(x_2, dx_2, s)) =: g_N(s, |x_1 - x_2|_1)dx_1 dx_2$$

decays as $|x_1 - x_2|_1$ goes to infinity.

With this question in mind we remark that $\rho_{2+m,2}(x_1, x_2, \dots x_{2+m})$ is the sum of at most $(2+m)!$ products

$$(-1)^\sigma \prod_{i=1}^{m+2} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

each containing at least two factors

$$\frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

with $x_i, x_{\sigma(i)}$ belonging to different intervals $[x_1, x_1 + s], [x_2, x_2 + s]$. Thus

$$(31)\qquad |\rho_{2+m,2}(\bar{x})| \leq (2+m)! \frac{\mathrm{const}(s)}{1 + |x_1 - x_2|_1^2}.$$

Here and further in our calculations we use different constants, depending on s (but not on N). Usually we will denote all of them const(s). The only property which we need from these constants is their uniform boundedness on every finite interval $s \in [0, T]$. Now (30) and (31) give us the desired estimate of $g_N(s, x)$:

$$\begin{aligned}
(32)\qquad |g_N(s,x)| &\leq \sum_{m=0}^{\infty} \frac{1}{m!}(2s)^m \min\{2, \mathrm{const}(s)(2+m)!/(1+|x|_1^2)\} \\
&\leq \sum_{0 \leq m \leq \mathrm{const}_1(s) \log x / \log(\log x)} \mathrm{const}(s)^m/(1+x^2) \\
&\quad + \sum_{m > \mathrm{const}_1(s) \log x / \log(\log x)} 2(2s)^m/m! \\
&\leq \mathrm{const}(s, \varepsilon)/(1 + |x|_1^{2-\varepsilon})
\end{aligned}$$

for any $\varepsilon > 0$. As N tends to infinity, $g_N(s, x)$ converges to the limit, uniformly in x:

$$g(s,x) := \lim_{N\to\infty} g_N(s,x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int \rho^{(\infty)}_{2+m,2}(0, x; \dots x_{m+2}) dx_3 \dots dx_{m+2}$$

where the variables $x_3, \dots x_{2+m}$ are integrated over $[0,s] \cup [x, x+s]$ and $\rho^{(\infty)}_{2+m,2}$, $m \geq 0$, is as defined in (26) with

$$\rho_n^{(\infty)}(y_1, \dots y_n) = \det\left(\frac{\sin \pi(y_i - y_j)}{\pi(y_i - y_j)}\right)_{i,j=1,\dots n}$$

being the n-point correlation functions in URML (7). Estimate (32) holds for $g(s,x)$ as well:

$$|g(s,x)| \leq \mathrm{const}(s,\varepsilon)/(1+|x|^{2-\varepsilon}).$$

Now we are in a position to write down the formula for the variance of $\eta_N(s)$:

$$\begin{aligned}
(33) \qquad \mathrm{Var}\, \eta_N(s) &= \mathbb{E}\Big(\int_0^N \big(\chi(x_1, dx_1, s) - \mathbb{E}\chi(x_1, dx_1, s)\big) \\
&\quad \cdot \int_0^N \big(\chi(x_2, dx_2, s) - \mathbb{E}\chi(x_2, dx_2, s)\big)\Big) \\
&= \mathbb{E}\int_0^N \underset{|x_1-x_2|_1>s}{\int_0^N} (\chi(x_1, dx_1, s) \\
&\quad - \mathbb{E}\chi(x_1, dx_1, s))(\chi(x_2, dx_2, s) - \mathbb{E}\chi(x_2, dx_2, s)) \\
&\quad + \mathbb{E}\int_0^N \underset{0<|x_1-x_2|_1\leq s}{\int_0^N} (\chi(x_1, dx_1, s) \\
&\quad - \mathbb{E}\chi(x_1, dx_1, s))(\chi(x_2, dx_2, s) - \mathbb{E}\chi(x_2, dx_2, s)) \\
&\quad + \mathbb{E}\int_0^N \underset{x_1=x_2}{\int_0^N} (\chi(x_1, dx_1, s) \\
&\quad - \mathbb{E}\chi(x_1, dx_1, s))(\chi(x_2, dx_2, s) - \mathbb{E}\chi(x_2, dx_2, s)) \\
&= \int_0^N \underset{|x_1-x_2|_1>s}{\int_0^N} g_N(s, |x_1 - x_2|_1) dx_1 dx_2 \\
&\quad - \int_0^N \underset{0<|x_1-x_2|_1\leq s}{\int_0^N} F_N(s)^2 dx_1 dx_2 \\
&\quad + \int_0^N \int_0^N F_N(s)\delta(x_1 - x_2) dx_1 dx_2 \\
&= b_N(s,s)N + O(N^{\varepsilon}),
\end{aligned}$$

where

$$b_N(s,s) = \int\limits_{|x|>s} g_N(s,x)dx - 2sF_N^2(s) + F_N(s). \tag{34}$$

Similar calculations give us the formula for the covariance of $\eta_N(s), \eta_N(t)$:

$$\mathrm{Cov}(\eta_N(s), \eta_N(t)) = b_N(s,t)N + O(N^{\varepsilon})$$

with

$$\begin{aligned} b_N(s,t) &= \int_{-\infty}^{-t} g_N(s,t,x)dx + \int_s^{+\infty} g_N(s,t,x)dx \\ &\quad - (s+t)F_N(s)F_N(t) + F_N(s\vee t), \end{aligned} \tag{35}$$

where the function $g_N(s,t,x)$ is defined as

$$g_N(s,t,x) = \sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int \rho_{2+m,2}^{(N)}(0,x;\ldots x_{m+2})dx_3\ldots dx_{m+2}$$

the variables $x_3,\ldots x_{2+m}$ are integrated over $[0,s]\cup[x,x+t]$, and $s\vee t := \max(s,t)$. Similar to (32)

$$|g_N(s,t,x)| \le \mathrm{const}(s,t,\varepsilon)/(1+|x|_1^{2-\varepsilon}) \quad \text{for any} \quad \varepsilon > 0. \tag{36}$$

As $N\to\infty$, $g_N(s,t,x)$ converges uniformly in x to the limit

$$\begin{aligned} g(s,t,x) &= \lim_{N\to\infty} g_N(s,t,x) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int \rho_{2+m,2}^{(\infty)}(0,x;\ldots x_{m+2})dx_3\ldots dx_{m+2}, \end{aligned} \tag{37}$$

where the variables $x_3,\ldots x_{2+m}$ are integrated over $[0,s]\cup[x,x+t]$.

The covariance function $b(s,t)$ of the limiting Gaussian process $\xi(s)$ is defined as

$$\begin{aligned} b(s,t) &= \lim_{N\to\infty} b_N(s,t) = \int_s^{+\infty} g(s,t,x)dx \\ &\quad + \int_{-\infty}^{-t} g(s,t,x)dx - (s+t)F(s)F(t) + F(s\vee t). \end{aligned} \tag{38}$$

It is a matter of lengthy, but simple, calculations to show that at the origin

$$\begin{aligned} b(s,s) &= \mathrm{Var}\ \xi(s) = \pi^2 s^3/9 + O(s^4), \\ F(s) &= 1 - \pi^2 s^3/9 + O(s^4). \end{aligned}$$

The functions $F(s), b(s,t) - F(s\vee t)$ are analytic, which implies

$$\mathbb{E}(\theta(s+\delta s) - \theta(s))^2 = O(\delta s)$$

and Hölder continuity of the random process $\xi(s)$ with any exponent less than 1/2.

To proceed with the proof of Theorem 1, we need the formulas for

$$\mathbb{E}\prod_1^{2k}(\chi(x_i,dx_i,s)-\mathbb{E}\chi(x_i,dx_i,s)),$$

similar to (26),(27). We will use again the special cluster structure of n-point correlation functions (21). In this way we will be able to prove that

$$\begin{aligned}(39)\quad &\mathbb{E}\prod_1^{2k}(\chi(x_i,dx_i,s)-\mathbb{E}\chi(x_i,dx_i,s))\\ &=\sum\prod_{(i,j)}\operatorname{Cov}(\chi(x_i,dx_i,s),\chi(x_j,dx_j,s))+\ R_{2k}(x_1,\dots x_{2k})dx_1\dots dx_{2k},\end{aligned}$$

where the summation $\sum$ is over all partitions of $\{1,\dots 2k\}$ into pairs (i,j) and for any $\varepsilon>0$

$$(40)\qquad \int_0^N\cdots\int_0^N|R_{2k}(x_1\dots x_{2k})|dx_1\dots dx_{2k}=O(N^{k-1+\varepsilon}).$$

Formulas (39), (40) are the key ingredients in the proof of Theorem 1.1.

Again, we will consider the contributions to $\mathbb{E}(\eta_N(s)-\mathbb{E}\eta_N(s))^{2k}$ from the "off-diagonal" terms

$$(41)\qquad \min_{i\neq j}|x_i-x_j|_1>s,$$

"near-diagonal" $0<\min_{i\neq j}|x_i-x_j|_1\le s$, and diagonal terms $x_i=x_j$ separately. In calculations to follow we restrict ourselves to the case of even moments. However, all arguments work in the case of odd moments as well. Let us consider first the "off-diagonal" case (41). By the inclusion-exclusion principle

$$\begin{aligned}(42)\quad &\mathbb{E}\prod_1^{2k}\chi(x_i,dx_i,s)\\ &=\left(\sum_{m=0}^{N-2k}\frac{(-1)^m}{m!}\int\int\rho_{2k+m}^{(N)}(x_1,\dots x_{2k};\dots x_{2k+m})dx_{2k+1}\dots dx_{2k+m}\right)dx_1\dots dx_{2k},\end{aligned}$$

the integral is over $(\bigcup_1^{2k}[x_i,x_i+s])^m$ and

$$(43)\qquad \rho_{2k+m}^{(N)}(\bar{x})=\sum_{\sigma\in S_{2k+m}}(-1)^\sigma\prod_{i=1}^{2k+m}\frac{\sin\pi(x_i-x_{\sigma(i)})}{N\sin(\pi(x_i-x_{\sigma(i)})/N)}.$$

As in (26) we decompose the sum (43) into two: the first subsum corresponds to the interaction between the particles $x_1,\dots,x_{2k}$, where each particle interacts with at least one other particle. The formal definition of the first subsum is the following: Denote by $x_{i_1}^{(j)},\dots x_{i_{p_j}}^{(j)}$ the variables among $x_1,\dots,x_{2k+m}$ belonging to the interval $[x_j,x_j+s], j=1,\dots 2k$.

We will also reserve the notations $\bar{x}^{(j)}$ for the vector $(x^{(j)}_{i_1}, \ldots x^{(j)}_{i_{p_j}})$ and $n(x_j) = p_j$ for the number of variables belonging to $[x_j, x_j + s]$. We define the first subsum as the sum over $\sigma \in S_{2k+m}$, such that for any $j = 1, \ldots 2k$ (i.e. for any particle x_j) there exists another index $1 \le l \le 2k$ (there exists another particle x_l), such that

$$\sigma(\{i^{(j)}_1, \ldots i^{(j)}_{p_j}\}) \cap \{i^{(l)}_1, \ldots i^{(l)}_{p_l}\} \neq \emptyset \tag{44}$$

(particles x_j and x_l interact with each other). We denote this sum by $\rho_{2k+m,2k}$.

To deal with the second sum, we single out the particles not interacting with any others. Iterating, we arrive at the formula:

(45)

$$\begin{aligned}\rho_{2k+m}(x_1, \ldots x_{2k}, \ldots x_{2k+m}) &= \rho_{2k+m,2k}(x_1, \ldots x_{2k}, \ldots x_{2k+m}) \\ &+ \sum_{\emptyset \neq \mathcal{A} \subset \{1,\ldots 2k\}} \left(\prod_{j \in \mathcal{A}} \rho_{n(x_j)}(\bar{x}^{(j)}) \right) \\ &\times \rho_{\,2k+m-\sum_{j\in\mathcal{A}} n(x_j),\ 2k-|\mathcal{A}|} \left(\bar{x} \setminus \cup_{j \in \mathcal{A}} \bar{x}^{(j)} \right).\end{aligned}$$

Since the following formula

$$\mathbb{E} \prod_1^{2k} \nu_i = \sum_{\mathcal{A} \subset \{1,\ldots 2k\}} \prod_{j \in \mathcal{A}} \mathbb{E}\ \nu_i \ \cdot\ \mathbb{E} \prod_{l \notin \mathcal{A}} (\nu_l - \mathbb{E}\ \nu_l)$$

is valid for arbitrary random variables ν_i, (23), (42) and (45) imply:

PROPOSITION 3.1. *Let* $\min_{i\neq j}|x_i - x_j| > s$, *and* $s \le (\log N)^{1/2}$. *Then*

$$\begin{aligned}\mathbb{E} \prod_1^{2k} &(\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)) = dx_1 \cdots dx_{2k} \\ &\times \left(\sum_{m=0}^{N-2k} \frac{(-1)^m}{m!} \int \rho^{(N)}_{2k+m,2k}(x_1, \ldots x_{2k}; \ldots x_{2k+m}) dx_{2k+1} \ldots dx_{2k+m} \right. \\ &+ \left. O\Big(\Big(\frac{\text{const}(s) \log^k N}{N}\Big)^N \cdot N^{2k}\Big) \right);\end{aligned} \tag{46}$$

the variables $x_{2k+1}, \ldots x_{2k+m}$ *are integrated over* $\left(\bigsqcup_1^{2k} [x_j, x_j + s] \right)^m$.

Remark. The remainder term is of the same nature as in (27) and is treated similarly.

Proposition 3.1 will play the central role in our proof, leading to (39), (40). To make our arguments clearer, we associate with any permutation $\sigma \in S_{2k+m}$ an oriented graph $\mathcal{J}(\sigma)$. By definition, the vertices of $J(\sigma)$ are integers $1, \ldots 2k$

(particles $x_1, \dots x_{2k}$) and there is a directed bond from the j^{th} particle to the l^{th} particle, $l \neq j$, if and only if (44) is satisfied. Then in our notation

$$\rho^{(N)}_{2k+m,2k}(\bar{x}) = \sum^{*}_{\sigma \in S_{2k+m}} (-1)^{\sigma} \prod_{i=1}^{2k+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

where $\overset{*}{\sum}$ is the sum over such σ that any maximal connected component of $j(\sigma)$ has at least two elements. For future consideration it is also useful to define $\tilde{\rho}_{2k+m,2k}(x_1, \dots x_{2k}, \dots x_{2k+m})$ as the sum over σ for which $\mathcal{J}(\sigma)$ is connected. We claim that the main contribution to (46) comes from the interaction between the pairs of particles. Representing $\mathcal{J}(\sigma)$ as a disjoint union of maximal connected components

$$\mathcal{A}_1, \dots \mathcal{A}_p : \bigsqcup_1^p \mathcal{A}_q = \{1, \dots 2k\}$$

and denoting $(x_i)_{i \in \mathcal{A}_q}$ by $\bar{x}(\mathcal{A}_q)$ we obtain the representation of $\rho_{2k+m,2k}(\bar{x})$ as the sum of products

$$\sum_{\substack{(\mathcal{A}_1, \dots \mathcal{A}_p) \\ |\mathcal{A}_q| \geq 2,\ q=1,\dots p}} \prod_1^p \tilde{\rho}_{\sum_{j \in \mathcal{A}_q} n(x_j),\ |\mathcal{A}_q|} \Big(\bar{x}\big(\mathcal{A}_q\big)\Big). \tag{47}$$

Now (46) and (47) give us

$$\begin{aligned} \mathbb{E} \prod_1^{2k}(\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)) = \prod_1^{2k} dx_i \Bigg(\sum^{**}_{\substack{(\mathcal{A}_1, \dots \mathcal{A}_p) \\ |\mathcal{A}_q| \geq 2, q=1,\dots p}} \prod_1^p \\ \left(\sum_{m=0}^{N-|\mathcal{A}_q|} \frac{(-1)^m}{m!} \int_{(\sqcup_{j \in \mathcal{A}_q}[x_j, x_j+s])^m} \tilde{\rho}_{|\mathcal{A}_q|+m, |\mathcal{A}_q|}(\bar{x}(\mathcal{A}_q), y_1, \dots y_m) dy_1 \dots y_m \right) \\ + O\Big(\big(\frac{\text{const}(s) \log^k N}{N}\big)^N \cdot N^{2k}\Big)\Bigg) \end{aligned} \tag{48}$$

where the sum $\overset{**}{\sum}$ is over all partitions of $\{1, \dots 2k\}$ into two- (or more) element subsets.

Since $\tilde{\rho}_{2+m,2}(\bar{x}) = \rho_{2+m,2}(\bar{x})$ the sum over partitions into the two-element subsets is exactly

$$\sum_{\substack{\text{partitions} \\ \text{into pairs}}} \prod_{(i,j)} \text{Cov}\Big(\chi(x_i, dx_i, s), \chi(x_j, dx_j, s)\Big). \tag{49}$$

To estimate the remaining part, let us introduce the notation

$$(50)\qquad r_{|\mathcal{A}_q|}(\bar{x}(\mathcal{A}_q), s) := \sum_{m=0}^{N-|\mathcal{A}_q|} \frac{(-1)^m}{m!} \times \int_{(\sqcup_{j\in\mathcal{A}_q}[x_j, x_j+s])^m} \tilde{\rho}_{|\mathcal{A}_q|+m, |\mathcal{A}_q|}(\bar{x}(\mathcal{A}_q), y_1, \dots y_m).dy_1 \dots y_m.$$

The next lemma, together with (48) clearly implies (39), (40):

LEMMA 3.2.

$$(51)\qquad \int_{\substack{[0,N]^l \\ \min_{i\neq j}|x_i - x_j|_1 > s}} r_l(x_1, \dots x_l, s) dx_1 \dots dx_l$$

$$= \begin{cases} \int_{|x|>s} g_N(s,x)dx \cdot N + o(N^\varepsilon) & \text{if } l = 2, \\ o(N^{1+\varepsilon}) & \text{if } l > 2 \end{cases}$$

for any $\varepsilon > 0$.

Remark. One can see from (48) that $r_l(x_1, \dots x_l, s)$ are Ursell functions of the s-modified random-point field. Compare (51) with the conditions on U_l in Theorem 2.1.

Proof of Lemma 3.2. The case $l = 2$ was considered above when we calculated the variance of $\eta_N(s)$.

Assume now $l > 2$ and denote by $r_{l,m}$ the m[th] term in (50). We are looking for the estimates on $\tilde{\rho}_{l+m,l}$ similar to (30), (31). Since $\tilde{\rho}_{l+m,l}$ can be obtained by the finite number of additions, subtractions and multiplications of

$$\det\left(\frac{\sin \pi(x_i - x_j)}{N \sin(\pi(x_i - x_j)/N)}\right),$$

(29) provides the estimates

$$(52)\qquad |\tilde{\rho}_{l+m,l}(x_1, \dots x_l, \dots x_{l+m})| < \text{const}_l$$

and

$$(53)\qquad \int_{\substack{[0,N]^l \\ \min_{i\neq j}|x_i - x_j|_1 > s}} |r_{l,m}(x_1, \dots x_l)|\, dx_1 \dots dx_l \;\leq\; \text{const}_l \frac{(l\ s)^m}{m!} N^l.$$

To get an estimate similar to (31), we write by definition:

$$\tilde{\rho}^{(N)}_{l+m,l}(x_1, \dots x_l, \dots x_{l+m}) = \sum_{\substack{\sigma\in S_{l+m} \\ \mathcal{J}(\sigma) \text{ is connected}}} (-1)^\sigma \prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}.$$

Consider an arbitrary term from this sum. Our goal is to estimate the absolute value of

$$\prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

by

$$\text{const}^{l+m}(s) \prod_1^l \frac{2}{1 + |x_j - x_{\tau(j)}|} \tag{54}$$

where τ is some cyclic permutation of integers $1, \ldots l$ (particles $x_1, \ldots x_l$), depending on σ and partition (44). To do this we will replace

$$\frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

by 1 whenever $x_i, x_{\sigma(i)}$ belong to the same segment $[x_j, x_j + s], j = 1, \ldots l$, and we will replace it by $2/(1 + |x_i - x_{\sigma(i)}|)$ in the opposite case.

If we write σ as a product of disjoint cyclic permutations

$$\sigma = \sigma_1 \cdot \ldots \sigma_m,$$

each $\sigma_p, p = 1, \ldots m$ determines some cyclic excursion on the graph $\mathcal{J}(\sigma)$; the steps of the excursion correspond to the terms $2/(1 + |x_i - x_{\sigma(i)}|)$ in our estimate.

Since the graph $\mathcal{J}(\sigma)$ is connected, the path of every excursion intersects the path of some other excursion, and after several switches we can go from one path to another (otherwise we would have a nontrivial maximal connected component of $\mathcal{J}$). Therefore we can combine these paths into one big cyclic path (with possible self-intersections), along which are all vertices of $\mathcal{J}$. Again, to each step of the path, $j \longrightarrow l$, there corresponds a term

$$\frac{2}{1 + |x_i - x_{\sigma(i)}|} : x_i \in [x_j, x_j + s], \quad x_{\sigma(i)} \in [x_l, x_l + s]$$

in our estimate.

The whole number of steps of the constructed path is $l+m$ at most[3]. Now we will eliminate all possible self-intersections. If j is the current position of our walk, n is the previous one and l is the next one, and the vertex j has already been visited, we replace two steps $n \longrightarrow j, j \longrightarrow l$ by one $n \longrightarrow l$ in the new, modified walk.

Using the inequalities

$$2/(1 + |x_i - x_{\sigma(i)}|) \leq \text{const}(s) 2/(1 + |x_j - x_l|),$$
$$2/(1 + |x - y|) \cdot 2/(1 + |y - z|) \leq 2 \cdot 2/(1 + |x - z|)$$

[3]See remark after the end of proof of Lemma 3.2.

and subsequently eliminating from the path the sites visited before, we finally obtain the path without self-intersections, which is given by some cyclic permutation $\tau \in S_l$. This leads to (54) and the inequality

$$\int_{(\sqcup_{j=1}^{l}[x_j, x_j+s])^m} \left| \prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)} \right| dx_{l+1} \dots dx_{l+m}$$

$$\leq (l \cdot s)^m \Big(\mathrm{const}(s)\Big)^{l+m} \sum_{\tau \in S_l} \prod_{1}^{l} \frac{2}{1 + |x_j - x_{\tau(j)}|}.$$

The inequality

$$\int_0^N 2/(1 + |x-y|) \cdot 2/(1 + |y - z|) dy \leq \mathrm{const} \log(N+1) 2/(1 + |x - z|)$$

implies

$$\int_{[0,N]^l} \prod_{j=1}^{l} 2/(1 + |x_j - x_{\tau(j)}|) dx_1 \dots dx_l \leq \mathrm{const} N \log^{l-2}(N+1)$$

and

$$(55) \qquad \left| \int_{[0,N]^l} r_{l,m}(x_1, \dots x_l) dx_1 \dots dx_l \right|$$

$$\leq \frac{1}{m!} l!(l+m)!(l\ s)^m (\mathrm{const}(s))^{l+m} N \log^{l-2}(N+1).$$

Finally, to get an estimate (51) on

$$\left| \int_{[0,N]^l} r_l(x_1, \dots x_l) dx_1 \dots dx_l \right| \leq \sum_0^\infty \int_{[0,N]^l} |r_l(x_1, \dots x_l)| dx_1 \dots dx_l$$

one can use (55) for $m < \mathrm{const}(s, l, \varepsilon) \log(N+1)$, where $\mathrm{const}(s, l, \varepsilon)$ is small enough, and (53) for $m \geq \mathrm{const}(s, l, \varepsilon) \log(N+1)$. □

Remark. The trivial bound for the number of steps of the path (at most $l+m$) is enough for our purposes now. However in the proof of Corollary 1.5 we will need an estimate that the number of steps is bounded by some number, depending only on l. To accomplish this we have to eliminate some loops of the path (i.e. replace the corresponding multipliers by 1). Namely we eliminate a loop, if after throwing it out, we still have a closed path, visiting all vertices of $\mathcal{J}$. After such a procedure is completed we arrive at the path with at most $\sum_1^{l-1} j + l - 1 = l(l+1)/2 - 1$ steps. Formulas (48), (49), (51) give the following result:

$$\mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \min_{i\neq j}|x_i - x_j|_1 > s}} \prod_1^{2k} (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)) \tag{56}$$
$$= (2k-1)!! \left(\int\limits_{|x|>s} g_N(s,x)dx \right)^k N^k + O\left(\mathrm{const}(s,\varepsilon) N^{k-1+\varepsilon}\right).$$

In the second part of the proof of Theorem 1.1 we take into account the contributions to $\mathbb{E}\Big(\eta_N(s) - \mathbb{E}\eta_N(s)\Big)^{2k}$ from the diagonal ($x_i = x_j$) and "near-diagonal" ($0 < |x_i - x_j|_1 \leq s$) terms. We introduce an equivalence relation on the set of particles $\{x_1, \ldots x_{2k}\}$, calling x_i, x_j the "neighbors," if there is a sequence of particles

$$x_{i_0} = x_i, x_{i_1}, \ldots x_{i_p} = x_j$$

such that

$$\max_{r=0,\ldots p-1} |x_{i_{r+1}} - x_{i_r}|_1 \leq s. \tag{57}$$

We claim that the contributions to $\mathbb{E}\Big(\eta_N(s) - \mathbb{E}\eta_N(s)\Big)^{2k}$ of order of N^k appear only when each equivalence class of (57) contains one or two particles. Assume that we have l two-element equivalence classes, say

$$\{x_1, x_2\}, \ldots \{x_{2l-1}, x_{2l}\}$$

and $2k - 2l$ one-element equivalence classes $\{x_{2l+1}\}, \ldots \{x_{2k}\}$. Since

$$\chi(x_i, dx_i, s)\, \chi(x_j, dx_j, s) = 0$$

if $0 < |x_i - x_j|_1 \leq s$, and always $\chi^2(x_i, dx_i, s) = \chi(x_i, dx_i, s)$, we have

$$\mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \{x_1,x_2\},\ldots\{x_{2l-1},x_{2l}\} \\ \{x_{2l+1}\},\ldots\{x_{2k}\}}} \prod_1^{2k} (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s))$$
$$= \mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \{x_1,x_2\},\ldots\{x_{2l-1},x_{2l}\} \\ \{x_{2l+1}\},\ldots\{x_{2k}\}}} \prod_{i=1}^{l} \Big[-\mathbb{E}\chi(x_{2i-1}, dx_{2i-1}, s)\mathbb{E}\chi(x_{2i}, dx_{2i}, s)$$
$$- \Big(\chi(x_{2i-1}, dx_{2i-1}, s) - \mathbb{E}\chi(x_{2i-1}, dx_{2i-1}, s)\Big)\, \mathbb{E}\,\chi(x_{2i}, dx_{2i}, s)$$
$$- \Big(\chi(x_{2i}, dx_{2i}, s) - \mathbb{E}\chi(x_{2i}, dx_{2i}, s)\Big)\, \mathbb{E}\,\chi(x_{2i-1}, dx_{2i-1}, s)$$

$$+ \mathbb{E}\ \chi(x_{2i}, dx_{2i}, s)\ \delta(x_{2i-1} - x_{2i})\ dx_{2i-1}\Big] \cdot$$

$$\cdot \prod_{j=2l+1}^{2k} \Big(\chi(x_j, dx_j, s) - \mathbb{E}\chi(x_j, dx_j, s)\Big)$$

$$= \Big(F_N(s) - 2sF_N^2(s)\Big)^l (2k-2l-1)!! \left(\int_{|x|>s} g_N(s,x)dx \right)^{k-l} N^k$$

$$+ O\Big(\mathrm{const}(s) N^{k-1+\varepsilon}\Big).$$

All such choices of equivalence classes produce

(58)

$$\sum_{l=0}^{k} \frac{(2k)!}{(2l)!(2k-2l)!} \frac{(2l)!}{l!\ 2^l} \Big(F_N(s) - 2sF_N^2(s)\Big)^l \frac{(2k-2l)!}{(k-l)!2^{k-l}} \Big(\int_{|x|>s} g_N(s,x)dx \Big)^{k-l}$$

$$\cdot N^k + O\Big(\mathrm{const}(s,\varepsilon)N^{k-1+\varepsilon}\Big)$$

$$= (2k-1)!! \left(F_N(s) - 2sF_N^2(s) + \int_{|x|>s} g_N(s,x)dx \right)^k N^k + O\Big(\mathrm{const}(s,\varepsilon)N^{k-1+\varepsilon}\Big).$$

If some equivalence classes have three elements or more, the contributing terms to $\mathbb{E}\Big(\eta_N(s) - \mathbb{E}\eta_N(s)\Big)^{2k}$ will be bounded by some power (say l) of $F_N(s) \int_0^N 1dx$, multiplied by the n-dimensional integral ($n < 2k - 2l$)

$$\int_{\substack{[0,N]^{2k} \\ \min_{i\neq j} |x_i - x_j|_1 > s}} \left| \mathbb{E} \prod_1^n (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)) \right|$$

and multiplied by the areas of some polyhedrons of size s. From (39), (40) it follows that these terms are of order of $O(N^{k-1})$. Thus

$$\text{(59)} \quad \mathbb{E}\left((\eta_N(s) - \mathbb{E}\eta_N(s)) / N^{1/2} \right)^{2k} = (2k-1)!!\, (b_N(s,s))^k + O(N^{-1+\varepsilon})$$
$$= (2k-1)!!\ (b(s,s))^k + o(1).$$

Similar calculations yield

$$\mathbb{E}\left((\eta_N(s) - \mathbb{E}\eta_N(s)) / N^{1/2} \right)^{2k+1} = o(N^{-1/2+\varepsilon}).$$

The convergence of the mixed moments

$$\mathbb{E} \prod_1^n \Big((\eta_N(s_i) - \mathbb{E}\eta_N(s_i)) / N^{1/2} \Big)$$

to the moments of the Gaussian random process $\xi(s)$ can be proved in the same way. Since the convergence of all moments to the Gaussian ones implies the convergence of finite-dimensional distributions, Theorem 1.1 is proved. □

4. Proof of Theorem 1.2 and corollaries

We start with the proof of Theorem 1.2 . Since

$$\left|\tilde{\xi}_N(s) - \xi_N(s)\right| \le N^{-1/2}, \tag{60}$$

the finite-dimensional distributions of $\tilde{\xi}_N(\cdot)$ also converge to those of the limiting Gaussian process $\xi_N(\cdot)$ as $N \to \infty$, and for the functional convergence of probability distributions of $\tilde{\xi}_N(\cdot)$ on $C\,[0,\infty)$ it is enough to prove the tightness (relative compactness) of any sequence of distributions of $\tilde{\xi}_{N_n}(\cdot)$ on $C\,[0,T]$, T is arbitrarily large, as $N_n \to \infty$ ([3]).

Let us define for continuous function $f \in C[0,T]$ and $\delta > 0$, the modulus of continuity as

$$\omega_f(\delta) = \sup|f(s) - f(t)| : \ 0 \le s,t \le T, |s-t| < \delta.$$

The classical criterion of relative compactness ([3]) tells that the family $\{\mathcal{P}\}$ of probability measures on $C[0,T]$ is relatively compact if and only if:

(i) For each arbitrary small positive α there exists an $A(\alpha)$, such that

$$\mathcal{P}\{f : |f(0)| > A\} < \alpha \quad , \quad \text{for any } \mathcal{P}. \tag{61}$$

(ii) For each $\alpha, \beta > 0$ there exists some $\delta(\alpha,\beta)$ such that

$$\mathcal{P}\{f : \omega_f(\delta) > \beta\} < \alpha, \quad \text{for any } \mathcal{P}. \tag{62}$$

The results of O. Costin and J. Lebowitz ([8]) tell us that

$$\tilde{\xi}_N(0) = O\left((\log N/N)^{1/2}\right)$$

which gives us (i). To prove (ii) we need the following lemma:

LEMMA 4.1. *There exist some constants c_1, c_2 depending on $T > 0$, such that*

$$\begin{aligned}(63)\ \mathcal{P}_N &\left\{|\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| < c_1\left(|t-s|^{1/20} + N^{-1/4}\right), \forall t,s \in [0,T] : |t-s| < \delta\right\} \\ &> \ 1 - c_2(\delta^{4/5} + N^{-1/20}\log N).\end{aligned}$$

Assuming Lemma 4.1 is proven, we can quickly finish the proof of Theorem 1.2. Choose N_*, δ_* such that

$$c_1 N_*^{-1/4} < \beta/2 \quad , \quad c_1\delta_*^{1/20} < \beta/2,$$
$$c_2 \log N_* N_*^{-1/20} < \alpha/2 \quad , \quad c_2\delta_*^{4/5} < \alpha/2.$$

For any fixed probability distribution $\mathcal{P}$ on $C[0,T]$ and arbitrary $\alpha, \beta > 0$ one can find some $\delta(\alpha,\beta,\mathcal{P}) > 0$ such that

$$\mathcal{P}\{f : \omega_f(\delta) > \beta\} < \alpha.$$

Let us choose such a δ for any $\tilde{\xi}_{N_i}, N_i < N_*$, and define the final δ as the minimum of such δ's and δ_*. Condition (ii) is satisfied; therefore the family of probability distributions given by $\tilde{\xi}_{N_n}(\cdot)$ is tight. Theorem 1.2 is proven. Now we shall prove Lemma 4.1.

First, let us note that if t, s belong to some interval of length $c_3 N^{-3/4}$, $c_3 \le 1: \ s,t \in [s', s' + c_3 N^{-3/4}]$ then

$$|\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| \le |\tilde{\xi}_N(s' + c_3 N^{-3/4}) - \tilde{\xi}_N(s')| + \mathrm{const}\, N^{-1/4}. \tag{64}$$

Indeed, by definition, for $s' \le s \le t \le s' + c_3 N^{-1/4}$

$$\begin{aligned} 0 &\le \tilde{\xi}_N(t) - \tilde{\xi}_N(s) + F_N(t)\, N^{1/2} - F_N(s)\, N^{1/2} \\ &\le \tilde{\xi}_N(s' + c_3 N^{-3/4}) - \tilde{\xi}_N(s') + F_N(s' + c_3 N^{-3/4})\, N^{1/2} - F_N(s')\, N^{1/2} \end{aligned}$$

which implies

$$\begin{aligned} |\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| &\le |\tilde{\xi}_N(s' + c_3 N^{-3/4}) - \tilde{\xi}_N(s')| \\ &\quad + N^{1/2} \cdot 2 \ \mathrm{Variation}_{[s', s' + c_3 N^{-3/4}]}\Big(F_N(s)\Big). \end{aligned}$$

The functions $F_N(s)$ are continuously differentiable uniformly in s on any finite interval. Indeed

$$\begin{aligned} (d/ds)F_N(s) &= \rho_2^{(N)}(0,s) - \int_0^s \rho_3^{(N)}(0,s,x_3)dx_3 \\ &\quad + \frac{1}{2!}\int_0^s \int_0^s \rho_4^{(N)}(0,s,x_3,x_4)dx_3dx_4 - \dots \\ &\le \sum_{k=0}^{\infty} \frac{1}{k!} s^k = \exp(s). \end{aligned}$$

This proves (64). Now we divide the segment $[0,T]$ into 2^k disjoint subsegments

$$\Delta_l^{(k)} = [l\, T/2^k, (l+1)T/2^k], \qquad l = 0, 1, \dots 2^k - 1$$

with k ranging from 1 to $\left[(\log T + \frac{3}{4}\log N)/\log 2\right] + 1$ (i.e. the length of $\Delta_l^{(k)}$ is always greater than $\frac{1}{2}N^{-3/4}$). Using the Chebyshev inequality and (59) we obtain:

$$\begin{aligned} \mathcal{P}_N\left(|\xi_N(t) - \xi_N(s)| > |t-s|^{1/20}\right) &\le \frac{\mathbb{E}|\xi_N(t) - \xi_N(s)|^4}{|t-s|^{1/5}} \\ &\le \frac{3\left(b_N(t,t) - b_N(s,t) - b_N(t,s) - b_N(s,s)\right)^2 + \mathrm{const}(T,\varepsilon)N^{-1+\varepsilon}}{|t-s|^{1/5}}. \end{aligned}$$

Covariance function $b_N(s,t)$ can be represented as the sum of two terms:

$$\begin{aligned} b_N(s,t) &= \left(\int_s^\infty g_n(s,t,x)dx + \int_{-\infty}^{-t} g_n(s,t,x)dx - (s+t)F_N(s)F_N(t)\right) \\ &\quad + F_N(s\vee t) \end{aligned}$$

where the partial derivatives of the first are uniformly bounded on any compact set (the proof is similar to that of the case of $F_n(s)$ since we have the estimates of the type (32) on $g_N(s,t,x)$ and $(\partial/\partial s)g_N(s,t,x), (\partial/\partial t)g_N(s,t,x)$.)
This implies

$$\mathcal{P}_N\left(|\xi_N(t)-\xi_N(s)| > |t-s|^{1/20}\right) \le \mathrm{const}(T)\left((t-s)^2 + N^{-19/20}\right)/|t-s|^{1/5}$$

where $t,s \in [0,T]$, and we choose $\varepsilon = 1/20$ which gives us (we denote all constants appearing in our calculations by $\mathrm{const}(T)$):

$$\begin{aligned} \mathcal{P}_N &\left(\bigsqcup_{l=0}^{2^k-1}\left\{\left|\xi_N((l+1)T/2^k) - \xi_N(l\;T/2^k)\right| > (T/2^k)^{1/20}\right\}\right) \\ &\le 2^k\,\mathrm{const}(T)\,\frac{(T/2^k)^2 + N^{-19/20}}{(T/2^k)^{1/5}} \\ &\le \mathrm{const}(T)\,\left((2^{-k})^{4/5} + N^{-19/20}(2^k)^{6/5}\right) \\ &\le \mathrm{const}(T)\,\left((2^{-k})^{4/5} + N^{-19/20}N^{18/20}\right) \\ &\le \mathrm{const}(T)\,\left((2^{-k})^{4/5} + N^{-1/20}\right), \end{aligned}$$

and

$$\begin{aligned} (65)\quad \mathcal{P}_N &\left(\bigsqcup_{k=k_0}^{[\frac{\log T + \frac{3}{4}\log N}{\log 2}]+1}\ \bigsqcup_{l=0}^{2^k-1}\left\{\left|\xi_N((l+1)T/2^k) - \xi_N(l\;T/2^N)\right| > (T/2^k)^{1/20}\right\}\right) \\ &\le \mathrm{const}(T)\left((2^{-k_0})^{4/5} + N^{-1/20}\log N\right). \end{aligned}$$

Choosing k_0 such that $2^{-k_0+1} \le \delta \le 2^{-k_0+2}$ and combining (65) with (64) and (60), we finish the proof. □

Corollary 1.3 can be proven by using the same machinery as in Theorem 1.1. To prove Corollary 1.4 one has to consider the continuous functional on $C[0,\infty)$:

$$G_T(f) = \sup_{s\in[0,T]}\left|f(s)\right|,$$

apply Theorem 1.2, and take into account that

$$\lim_{N\to\infty}\ \sup_{s\in[0,T]}\left|\tilde{\xi}_N(s) - \xi_N(s)\right| = 0.$$

Before we proceed with the proof of Corollary 1.5 we want to obtain an estimate on

$$\sup_{[0,\infty)} \left|\mathbb{E}\ \eta(I_N, s) - F(s)\ \frac{N\,|I_N|}{2\pi}\right|.$$

LEMMA 4.2.

$$\sup_{[0,\infty)} \left|\mathbb{E}\ \eta(I_N, s) - F(s)\ \frac{N|I_N|}{2\pi}\right| = o(N^{\varepsilon}\frac{|I_N|}{2\pi})$$

for any $\varepsilon > 0$.

Proof of 4.2. Assume first that

(66)

$$\begin{aligned} s \le (\log N)^{1/2} : \\ \Big|F_N(s) - F(s)\Big| &= \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \int_{[0,s]^n} \rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})\ d\bar{x} \\ &\quad + \sum_{n=N}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{[0,s]^n} \rho_{n+1}^{(\infty)}(0,\bar{x})\ d\bar{x} \\ &\le \sum_{n=0}^{N-1} \frac{1}{n!}\ \text{Volume}\,([0,s]^n) \sup_{\bar{x}\in[0,s]^n} \Big|\rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})\Big| \\ &\quad + \sum_{n=N}^{\infty} \frac{1}{n!}\ \text{Volume}\,([0,s]^n)\cdot 1. \end{aligned}$$

Using the inequality

$$\sup_{|x|\le s} \left|\frac{\sin \pi x}{N\sin(\pi x/N)} - \frac{\sin \pi x}{\pi x}\right| < \text{const}\ (s/N)^2$$

and the representation of $\rho_{n+1}^{(N)}$, $\rho_{n+1}^{(\infty)}$ as determinants (6), (7), we have

$$\sup_{\bar{x}\in[0,s]^n} \Big|\rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})\Big| \le \text{const}\ (n+1)!\,(n+1)\,(s/N)^2.$$

Since $\rho_{n+1}^{(N)}$, $\rho_{n+1}^{(\infty)}$ are not greater than 1 by absolute value, we finally arrive at

$$\begin{aligned} \Big|F_N(s) - F(s)\Big| &\le \sum_{n=0}^{N-1} \frac{s^n}{n!} \min\Big(2,\ \text{const}\ (n+1)!\,(n+1)\,(s/N)^2\Big) + \sum_{N}^{\infty} \frac{s^n}{n!} \\ &= o\Big((s/N)^{1-\varepsilon}\Big) + O(s^N/\,N!) \end{aligned}$$

for any $\varepsilon > 0$. Thus

$$\sup_{s\le(\log N)^{1/2}} \Big|F_N(s) - F(s)\Big| = o(N^{-1+\varepsilon}). \tag{67}$$

The function $F(s)$ decays at infinity superexponentially:

$$\log F(s) = -\pi^2 s^2/8 + O(s) \tag{68}$$

(see [22], [21], [26]), which gives us

$$F\Big((\log N)^{1/2}\Big) = o(N^{-9/8}). \tag{69}$$

Since $0 \le F_N(s) \le F_N\Big((\log N)^{1/2}\Big)$, $0 \le F(s) \le F\Big((\log N)^{1/2}\Big)$ for any $s \ge (\log N)^{1/2}$, (67) and (68) establish

$$\sup_{[0,\infty)} \Big|F_N(s) - F(s)\Big| = o(N^{-1+\varepsilon}) \tag{70}$$

$$\sup_{[0,\infty)} \Big|\mathbb{E}\ \eta(I_N, s) - F(s)\ \frac{N|I_N|}{2\pi}\Big| = o(N^{\varepsilon}\frac{|I_N|}{2\pi}). \qquad \square$$

We finish this section with the proof of Corollary 1.5.

Proof of Corollary 1.5. Since the tails of distribution functions $F(s)$, $F_N(s)$ are small enough (see (67), (69)), we have to prove only that

$$\sup_{0\le s\le(\log N)^{1/2}} \Big|\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)\Big| = o\left((N\frac{|I_N|}{2\pi})^{1/2\ +\varepsilon}\right).$$

To do this we will estimate the moments of $\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)$, when s is allowed to grow up slightly, $s \le (\log N)^{1/2}$:

LEMMA 4.3. *For any even integer $2k$, arbitrary small $\varepsilon > 0$ and $0 \le s \le (\log N)^{1/2}$ there exists some constant $c(\varepsilon, 2k)$, depending only on ε and $2k$, such that*

$$\mathbb{E}\ \left(\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)\right)^{2k}\ \le c(\varepsilon, 2k)\ \left(N\frac{|I_N|}{2\pi}\right)^{k+\varepsilon}.$$

Proof. Again, without loss in generality, we can assume I_N to be a unit circle, $I_N = [-\pi, \pi]$. Examinating the proof of Theorem 1.1 we realize that what is needed is the following generalization of estimates (51) from Lemma 3.2:

$$\sup_{0\le s\le(\log N)^{1/2}}\left| \int_{\substack{[0,N]^l \\ \min_{i\neq j}|x_i - x_j|_1 > s}} r_l(x_1, \dots x_l, s)dx_1 \dots dx_l\right| \tag{71}$$
$$\le \text{const}(\varepsilon, l)\ N^{l/2+\varepsilon}, l \ge 2.$$

Going along the lines of calculations from Theorem 1.1, we will clarify the dependence on s of the constants appearing there. We recall that we derived the formula for the Ursell functions r_l of the s-modified random field in (50) as

$$(72) \quad r_l(x_1, \dots x_l, s) = \sum_{m=0}^{N-l} \frac{(-1)^m}{m!} \int \tilde{\rho}^{(N)}_{l+m,l}(x_1, \dots x_l; \dots x_{l+m}) dx_{l+1} \dots dx_{l+m}$$

where the integration is over $\left(\bigsqcup_1^l [x_i, x_i + s]\right)^m$. For simplicity we will consider the cases $l = 2$ and $l > 2$ separately. Let us first take $l = 2$. Rewrite (72):

$$r_l(x_1, x_2, s) = \sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int_{(\bigsqcup_1^2 [x_i, x_i+s])^m} \rho^{(N)}_{2+m,2}(x_1, x_2, \dots x_{2+m}) dx_3 \dots dx_{2+m}.$$

Estimates (30), (31) give us

$$|\rho_{2+m,2}(\bar{x})| \leq \min\left(2,\ (m+2)! \frac{2}{1 + \max\left(|x_1 - x_2|_1 - s, 0\right)^2}\right).$$

Using the inequality

$$(73) \qquad 1/\left(1 + \max(x - s, 0)^2\right) \leq (2 + s^2)/(1 + x^2)$$

we obtain

$$\begin{aligned} |\rho_{2+m,2}(\bar{x})| &\leq \min\left(2; (m+2)! \frac{2(2+s^2)}{1 + |x_1 - x_2|_1^2}\right) \\ &\leq 2(2 + \log N) \min\left(1; \frac{(m+2)!}{1 + |x_1 - x_2|_1^2}\right), \\ |r_2(x_1, x_2)| &\leq \sum_{m=0}^{N-2} \frac{1}{m!} (2s)^m 2(2 + \log N) \min\left(1; \frac{(m+2)!}{1 + |x_1 - x_2|_1^2}\right). \end{aligned}$$

Fix some $\varepsilon > 0$. If $|x_1 - x_2|_1 < N^{\varepsilon/2}$, we arrive at

$$(74) \qquad \left|r_2(x_1, x_2)\right| \leq 2(2 + \log N)\exp(2s) = 2(2 + \log N)\exp\left(2(\log N)^{1/2}\right).$$

If $|x_1 - x_2|_1 \geq N^{\varepsilon/2}$, then for any $\varepsilon_1 > 0$

$$(2s)^m = o\left(|x_1 - x_2|_1^{1+\varepsilon_1}\right) \ \text{if} \ (m+2)! = O(1 + |x_1 - x_2|_1^2)$$

and

$$(75) \qquad \left|r_2(x_1, x_2)\right| \leq 2\ (2 + \log N)\mathrm{const}(\varepsilon) \left(\frac{1}{1 + |x_1 - x_2|_1^2}\right)^{1/2 - \varepsilon/4}$$

Integrating the right-hand side of (74), (75) over

$$\{x_1, x_2 \in [0, N], |x_1 - x_2| < (\geq) N^{\varepsilon/2}\}$$

we obtain the desired result. The case $l \geq 2$ will be treated in a similar fashion. Again the inequalities (52), (54) are crucial in our calculations. We keep the former and slightly refine the latter. Namely, using the procedure described on pages 21–22, Section 3 and the remark on page 23, we can estimate each term in $\tilde{\rho}_{l+m,l}$ as

$$\left|\prod_{1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}\right| \le \prod_{k=1}^{K} \frac{2}{1 + \max\left(|x_{j_{k+1}} - x_{j_k}|_1 - s, 0\right)}$$

where $\sigma \in S_{l+m}$ is such, that $\mathcal{J}(\sigma)$ is connected, and

$$j_1 \to j_2 \ldots \to j_K \to j_1$$

is some closed path on $\mathcal{J}(\sigma)$ with possible intersections, visiting all vertices $\{1, \ldots l\}$, with the number of steps K not greater than $l(l+1)/2 - 1$. Using the inequality (73) and eliminating possible self-intersections as explained in Section 3, we arrive at

$$\left|\prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}\right| \le \Big(2(2+s^2)\Big)^{(l+1)l/2-1} \prod_{j=1}^{l} \frac{2}{1 + |x_j - x_{\tau(j)}|_1} \tag{76}$$

where τ is some cyclic permutation of integers $1, \ldots l$. Estimates (52), (76) imply

$$|\tilde{\rho}_{l+m,l}| \le \sum_{\tau \in S_l} \min\left(\mathrm{const}(l), (m+l)!\Big(2(2+\log N)\Big)^{(l+1)l/2-1} \prod_{j=1}^{l} \frac{2}{1 + |x_j - x_{\tau(j)}|_1}\right)$$

and

(77)

$$\Big|r_l(x_1, \ldots x_l, s)\Big| \le \mathrm{const}(l)\Big(2\ (2+\log N)\Big)^{(l+1)l/2-1} \cdot \sum_{\tau \in S_l} \sum_{m=0}^{\infty} \frac{1}{m!}(l\ s)^m \min\left(1; (m+l)! \prod_{j=1}^{l} \frac{2}{1 + |x_j - x_{\tau(j)}|_1}\right).$$

Fix $\tau \in S_l$. If

$$\prod_{j=1}^{l}\Big(1 + |x_j - x_{\tau(j)}|_1\Big) < N^{\varepsilon/(2l)}, \tag{78}$$

the corresponding summand in (77) will be estimated from above by

$$\mathrm{const}(l)(4 + 2\log N)^{(l+1)l/2-1} \exp\Big(l(\log N)^{1/2}\Big). \tag{79}$$

If

$$\prod_{j=1}^{l}\Big(1 + |x_j - x_{\tau(j)}|_1\Big) \ge N^{\varepsilon/(2l)}, \tag{80}$$

then for any $\varepsilon_1 > 0$

$$(l,s)^m = o\Big(\prod_{j=1}^{l}\big(1+|x_j-x_{\tau(j)}|_1\big)^{1/2+\varepsilon_1}\Big), \text{if} (m+l)! = O\left(\prod_{j=1}^{l}\Big(\frac{2}{1+|x_j-x_{\tau(j)}|_1}\Big)\right)$$

and the corresponding summand is not greater than

$$\text{const}(l)(4+2\log N)^{(l+1)l/2-1}\text{const}(\varepsilon_1)\left(\prod_{j=1}^{l}\frac{1}{1+|x_j-x_{\tau(j)}|_1}\right)^{1/2-\varepsilon_1} \tag{81}$$

for any $\varepsilon_1 > 0$. Integrating (79), (81) over the domains (78), (80) we obtain the desired result. Lemma 4.3 is proved. □

Using Lemma 4.3 and the Chebyshev inequality we conclude that for any $n > 0, \varepsilon > 0$, there exists some constant, depending only on ε, n such that for any fixed $s, 0 \le s \le (\log N)^{1/2}$

$$\mathcal{P}_N\Big\{|\eta_N(s) - \mathbb{E}\eta_N(s)| > 1/3N^{1/2+\varepsilon}\Big\} < \text{const}(\varepsilon, n)N^{-n}. \tag{82}$$

Dividing the segment $[0,(\log N)^{1/2}]$ into $M = \Big[(\log N)^{1/2}N^{3/4}\Big]$ segments

$$[s_i, s_{i+1}], s_i = (\log N)^{1/2}i/M; i = 0,\dots M-1,$$

we estimate the probability

$$\mathcal{P}_N\Big\{\sup_{s_i}|\eta_N(s_i) - \mathbb{E}\eta_N(s_i)| > 1/2N^{1/2+\varepsilon}\Big\}$$

from above by the sum of probabilities:

$$\mathcal{P}_N\Big\{\sup_{s_i}|\eta_N(s_i) - \mathbb{E}\eta_N(s_i)| > 1/2N^{1/2+\varepsilon}\Big\} < \text{const}(\varepsilon, n)(\log N)^{1/2}N^{-n+3/4}. \tag{83}$$

To finish the proof we claim that variations of $\eta_N(s)$ on the segments $[s_i, s_{i+1}]$ are negligibly small, as well as the tails of $\eta_N(s)$ and $\mathbb{E}\eta_N(s)$ when $s \ge (\log N)^{1/2}$. Namely,

$$\begin{aligned} \text{Variation}_{[s_i,s_{i+1}]}\eta_N(s) &= \Big|\eta_N(s_{i+1}) - \eta_N(s_i)\Big| \\ &\le \Big|\eta_N(s_{i+1}) - \mathbb{E}\eta_N(s_{i+1})\Big| + \Big|\eta_N(s_i) - \mathbb{E}\eta_N(s_i)\Big| \\ &\quad + \text{Variation}_{[s_i,s_{i+1}]}\mathbb{E}\eta_N(s) \end{aligned} \tag{84}$$

and by the smoothness of $F(s)$,

$$\begin{aligned} \text{Variation}_{[s_i,s_{i+1}]}\mathbb{E}\eta_N(s) &\le \sup_{[0,(\log N)^{1/2}]}\Big|\mathbb{E}\eta_N(s) - NF(s)\Big| \\ &\quad + N\text{Variation}_{[s_i,s_{i+1}]}F(s) \\ &= o(N^{\varepsilon}) + O(N^{1/4}). \end{aligned} \tag{85}$$

Estimates (83), (84), (85) imply

$$\mathcal{P}_N\Big\{\sup_{0\le s\le(\log N)^{1/2}}\Big|\eta_N(s)-\mathbb{E}\eta_N(s)\Big| > N^{1/2+\varepsilon}\Big\} < \mathrm{const}(\varepsilon,n)N^{-n+3/4}(\log N)^{1/2}.$$

Choosing $n > 7/4$ we have

$$\sum_{N=1}^{\infty}\mathcal{P}\Big\{\sup_{[0,(\log N)^{1/2}]}\Big|\eta_N(s)-\mathbb{E}\eta_N(s)\Big| > N^{1/2+\varepsilon}\Big\} < \infty$$

and applying the Borel-Cantelli lemma ([13]) we find that with probability 1 there exists some integer N_0, such that for any $N > N_0$

$$\sup_{[0,(\log N)^{1/2}]}\Big|\eta_N(s)-\mathbb{E}\eta_N(s)\Big| \le N^{1/2+\varepsilon}. \tag{86}$$

Since for $s > (\log N)^{1/2}$

$$\begin{aligned}\eta_N(s) &\le \eta\Big((\log N)^{1/2}\Big),\\ \mathbb{E}\eta_N(s) &= NF_N(s) \le NF_N\Big((\log N)^{1/2}\Big),\\ NF(s) &\le NF\Big((\log N)^{1/2}\Big)\end{aligned}$$

and

$$\begin{aligned}&NF\Big((\log N)^{1/2}\Big) = o\Big(N^{-1/8}\Big),\\ &N\Big(F\Big((\log N)^{1/2}\Big) - F_N\Big((\log N)^{1/2}\Big)\Big) = o(N^{\varepsilon})\end{aligned}$$

(see (69), (70)), we can extend the sup in (86) to the whole real axis:

$$\sup_{[0,\infty)}\Big|\eta_N(s)-\mathbb{E}\eta_N(s)\Big| \le \mathrm{const}(\varepsilon)N^{1/2+\varepsilon}.$$

Corollary 1.5 is proved. □

5. Orthogonal and symplectic groups

The results formulated in Section 1 are valid for the other classical compact groups as well. The key factor here is the Vandermonde determinant nature of the density of the distribution function of eigenvalues. Formulas for the distribution of the eigenvalues with respect to the normalized Haar measure are classical (see ([35])). However it has been noted only recently by N. Katz and P. Sarnak ([19]) that the corresponding n-point correlation functions have the form of determinants, similar to (6). For the unitary group this fact was known for more than thirty years, back to pioneering papers by F. Dyson, M. Gaudin and M. L. Mehta ([30], [11]). Below we write the formulas for the distribution of the eigenvalues and n-point correlation functions for $SO(2N)$, $SO(2N+1)$, $USp(2N)$, $O_-(2N+2)$.

The SO($2N$) *case.* The eigenvalues of matrix M in SO($2N$) can be arranged in pairs:

$$\exp(i\theta_1), \exp(-i\theta_1), \ldots \exp(i\theta_N), \exp(-i\theta_N), \qquad (87)$$
$$0 \leq \theta_1 \leq \theta_2 \leq \ldots \theta_N \leq \pi.$$

The probability distribution of eigenvalues is defined by its density:

$$P_N(\theta_1, \ldots \theta_N) = 2\left(\frac{1}{2\pi}\right)^N \prod_{1 \leq i < j \leq N} (2\cos\theta_i - 2\cos\theta_j)^2. \qquad (88)$$

In the rescaled coordinates

$$x_i = (2N-1)\frac{\theta_i}{2\pi}, \quad 0 \leq x_1 \leq \ldots x_N \leq N - 1/2,$$

n-point correlation functions are equal to

$$\mathcal{R}_n^{(N)}(x_1, \ldots x_n) = \det\left(\frac{\sin\pi(x_i - x_j)}{(2N-1)\sin\Big(\pi(x_i - x_j)/(2N-1)\Big)} + \frac{\sin\pi(x_i + x_j)}{(2N-1)\sin\Big(\pi(x_i + x_j)/(2N-1)\Big)}\right)_{i,j=1,\ldots n} \qquad (89)$$

Note the similarity of (89) and (6). Since

$$\frac{\sin\pi(x_i + x_j)}{(2N-1)\sin\Big(\pi(x_i + x_j)/(2N-1)\Big)}$$

is small when $x_i, x_j \gg 1$, n-point correlation function (89) can be considered as a small perturbation of

$$\det\left(\frac{\sin\pi(x_i - x_j)}{(2N-1)\sin\Big(\pi(x_i - x_j)/(2N-1)\Big)}\right)_{i,j=1,\ldots n}. \qquad (90)$$

The SO($2N+1$) *case.* The first $2N$ eigenvalues of matrix M from SO($2N+1$) can be arranged in pairs as in (87). The $2N+1^{\text{th}}$ eigenvalue equals 1. The probability distribution of eigenvalues is defined by its density:

$$P_N(\theta_1, \ldots \theta_N) = (2/\pi)^N \prod_{1 \leq i < j \leq N} (2\cos\theta_i - 2\cos\theta_j)^2 \prod_{i=1}^{N} \sin^2(\theta_i/2). \qquad (91)$$

In the rescaled coordinates

$$x_i = N\theta_i\pi, \quad 0 \leq x_1 \leq \ldots x_N \leq N,$$

n-point correlation functions are given by the formula

$$(92)\quad \mathcal{R}_n^{(N)}(x_1,\dots x_n) = \det\left(\frac{\sin\pi(x_i-x_j)}{2N\sin(\pi(x_i-x_j)/2N)} - \frac{\sin\pi(x_i+x_j)}{2N\sin(\pi(x_i+x_j)/2N)}\right)_{i,j=1,\dots n}$$

The USp$(2N)$ *case.* The eigenvalues of matrix M in USp$(2N)$ can be arranged in pairs:

$$(93)\quad \begin{aligned}&\exp(i\theta_1),\exp(-i\theta_1),\dots\exp(i\theta_N),\exp(-i\theta_N),\\ &0\le\theta_1\le\theta_2\le\dots\theta_N\le\pi.\end{aligned}$$

The probability distribution of eigenvalues is defined by its density:

$$(94)\quad P_N(\theta_1,\dots\theta_N) = (2/\pi)^N \prod_{1\le i<j\le N}(2\cos\theta_i-2\cos\theta_j)^2\prod_{i=1}^N\sin^2(\theta_i).$$

In the rescaled coordinates

$$x_i=(2N+1)\theta_i/(2\pi),\quad 0\le x_1\le\dots x_N\le(2N+1)/2,$$

n-point correlation functions are equal to

$$(95)\quad \mathcal{R}_n^{(N)}(x_1,\dots x_n) = \det\left(\frac{\sin\pi(x_i-x_j)}{(2N+1)\sin\Big(\pi(x_i-x_j)/(2N+1)\Big)} - \frac{\sin\pi(x_i+x_j)}{(2N+1)\sin\Big(\pi(x_i+x_j)/(2N+1)\Big)}\right)_{i,j=1,\dots n}.$$

The O$_-(2N+2)$ *case.* The first $2N$ eigenvalues can be arranged in pairs, similar to (87); the $(2N+1)^{\text{th}}$ and $(2N+2)^{\text{th}}$ eigenvalues are $+1$ and -1. The formulas for $P_N(\theta_1,\dots\theta_N),\mathcal{R}_n^{(N)}(x_1,\dots x_n)$ coincide with those from the USp$(2N)$ case. The following universal result is valid for all cases, considered above.

Proposition 5.1. *Let I_N be an arbitrary subinterval of* $[0,\pi]\Big([-\pi,0]\Big)$, *such that the average number of eigenvalues hitting I_N tends to infinity (i.e. $N|I_N|/\pi\to\infty$). Then $\Big(\eta(I_N,s)-\mathbb{E}\eta(I_N,s)\Big)/\Big(N\|I_N|/\pi\Big)^{1/2}$ converges in finite-dimensional distributions to the Gaussian random process of Theorem* 1.1. *Theorem* 1.2 *and Corollaries* 1.3, 1.4, 1.5 *also hold.*

We have to examine two aspects of the proof of Theorem 1.1: combinatorial and analytical. Since n-point correlation functions (89), (92), (95) still have the form

$$\det\Big(K_N(x_i,x_j)\Big)_{i,j=1,\dots n},$$

all combinatorial considerations (for example formula (50), expressing Ursell functions of the s-modified random field through the n-point correlation functions of the original random-point field) remain the same. From the analytical point of view, we must treat

$$\begin{aligned}\mathcal{R}_n^{(N)}(x_1,\dots x_n) &= \det\left(\frac{\sin\pi(x_i-x_j)}{(2N+p)\sin(\pi(x_i-x_j)/(2N+p))}\right.\\ &\qquad \left.\pm\frac{\sin\pi(x_i+x_j)}{(2N+p)\sin(\pi(x_i+x_j)/(2N+p))}\right),\end{aligned}$$

$p=-1,0,1$, as a small perturbation of

$$\rho_n^{(2N+p)}(x_1,\dots x_n)=\det\left(\frac{\sin\pi(x_i-x_j)}{(2N+p)\sin(\pi(x_i-x_j)/(2N+p))}\right)_{i,j=1,\dots n}.$$

That is, if $x_2,\dots x_{1+m}\in[x_1,x_1+s]$, then

$$\Big|\mathcal{R}_{1+m}^{(N)}-\rho_{1+m}^{(2N+p)}\Big|\le\min\Big(2;(m+1)!(m+1)2/(1+|x_1|_1)\Big)$$

and

$$\begin{aligned}&\mathbb{E}\eta_N\Big([0,\pi],s\Big)\\ &=\int_0^{\frac{2N+p}{2}-s}dx_1\sum_{m=0}^{N-1}\frac{(-1)^m}{m!}\\ &\qquad\cdot\int_{[x_1,x_1+s]^m}\mathcal{R}_{1+m}^{(N)}(x_1,x_2,\dots x_{1+m})dx_2\dots dx_{1+m}\\ &\quad+\int_{\frac{2N+p}{2}-s}^{\frac{2N+p}{2}}dx_1\sum_{m=0}^{N-1}\frac{(-1)^m}{m!}\\ &\qquad\cdot\int_{[x_1,(2N+p)/2]^m}\mathcal{R}_{1+m}^{(N)}(x_1,x_2,\dots x_{1+m})\,dx_2\dots dx_{1+m}\\ &=\int_0^{\frac{2N+p}{2}}dx_1\sum_{m=0}^{N-1}\frac{(-1)^m}{m!}\int_{[x_1,x_1+s]^m}\rho_{1+m}^{(2N+p)}(x_1,x_2,\dots x_{1+m})dx_2\dots dx_{1+m}\\ &\quad+\text{Remainder term.}\end{aligned}$$

The remainder term can be estimated as

$$\begin{aligned}&\Big|\text{Remainder term}\Big| \\ &\le \int_0^{\frac{2N+p}{2}} dx_1 \sum_{m=0}^{N-1} \frac{1}{m!} s^m \min\Big(2;(m+1)!(m+1)2/(1+|x_1|_1)\Big) + \ s\exp(s)\end{aligned}$$

where $|x|_1 = \min(x, N+p/2-x)$, which implies

$$\Big|\mathbb{E}\eta_N\Big([o,\pi],s\Big) - NF_{2N+p}(s)\Big| \le \text{const}(s,\varepsilon)N^{\varepsilon}$$

for any $\varepsilon > 0$. Similarly to calculations in Section 4 one can show that

$$\sup_{[0,\infty)}\Big|\mathbb{E}\eta_N\Big([0,\pi],s\Big) - NF(s)\Big| = o(N^{1/2\varepsilon}).$$

Calculating the variance of $\eta_N\Big([0,\pi],s\Big)$ we note that if

$$x_3, \ldots x_{2+m} \in [x_1, x_1+s] \sqcup [x_2, x_2+s]$$

then

$$\begin{aligned}\Big|\mathcal{R}^{(N)}_{2+m,m} - \rho^{(2N+p)}_{2+m,2}\Big| \ &\le \ \min\Bigg(4;(m+2)!2^{m+2} \\ &\cdot \Big(2\Big(\frac{1}{1+\max(|x_1-x_2|-s;0)}\Big)\Big(\frac{1}{1+2|(x_1+x_2)/2|_1}\Big) \\ &+ \Big(\frac{1}{1+2|(x_1+x_2)/2|_1}\Big)^2\Big)\Bigg)\end{aligned}$$

and

$$\begin{aligned}&\Bigg|\sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int_{([x_1,x_1+s]\sqcup[x_2,x_2+s])^m} \mathcal{R}^{(N)}_{2+m,2}(x_1,x_2,\ldots x_{2+m}) \\ &\quad - \ \rho^{(2N+p)}_{2+m,2}(x_1,x_2,\ldots x_{2+m})dx_3\ldots x_{2+m}\Bigg| \\ &\le \ \text{const}(s,\varepsilon)\Big(1/(1+|x_1-x_2|)\Big)^{1-\varepsilon/2}\Big(1/(1+2|(x_1+x_2)/2|_1)\Big)^{1-\varepsilon/2}.\end{aligned}$$

The last inequality leads to the estimate

$$\text{Var }\eta_N\Big([0,\pi],s\Big) = b_{2N+p}(s,s)N + o(N^{\varepsilon})$$

valid for any $\varepsilon > 0$ and fixed s. The calculation of higher moments (i.e. the proof of Lemma 3.2 for $l > 2$) does not require any alterations. Since the distribution of the eigenvalues on $[-\pi, 0]$ is the mirror image of that on $[0,\pi]$,

$$\eta_N\Big([-\pi,\pi],s\Big) = 2\eta_N\Big([0,\pi],s\Big) - (0, 1 \text{ or } 2)$$

and $\Big(\eta_N\Big([-\pi,\pi],s\Big) - \mathbb{E}\eta_N\Big([-\pi,\pi],s\Big)\Big)/(2N)^{1/2}$ converges in finite-dimensional distributions to $2^{1/2}\xi(s)$. As soon as we prove Theorems 1.1, 1.2 for SO($2N+p$),

$(\mathrm{U}(N))$, the same results will hold for $\mathrm{O}(2N+p)$, $(\mathrm{SU}(N))$: Since $\eta_N\Big([-\pi,\pi],s\Big)$ is invariant under the matrix multiplication by -1 (or by $\exp(i\theta)$ in the unitary case),

$$\int\limits_{\mathrm{SO}(2N+p)} \eta_N^k\Big([-\pi,\pi],s\Big)d\mathrm{Haar}(\mathrm{SO}(2N+p)) = \int\limits_{\mathrm{O}(2N+p)} \eta_N^k\Big([-\pi,\pi],s\Big)d\mathrm{Haar}(\mathrm{O}(2N+p)),$$

$$\int\limits_{\mathrm{SU}(N)} \eta_N^k\Big([-\pi,\pi],s\Big)d\mathrm{Haar}(\mathrm{SU}(N)) = \int\limits_{\mathrm{U}(N)} \eta_N^k\Big([-\pi,\pi],s\Big)d\mathrm{Haar}(\mathrm{U}(N)).$$

Clearly, the analogues of Theorems 1.6, 1.7 are valid for the random-point field on the semi-axis $[0,\infty)$ with n-point correlation functions given by the formula

$$\rho_n(x_1,\dots x_n) = \det\left(\frac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)} \pm \frac{\sin\pi(x_i+x_j)}{\pi(x_i+x_j)}\right)_{i,j=1,\dots n}.$$

6. Circular orthogonal ensemble

The C.O.E. (log-gas (1) with the inverse temperature $\beta=1$) corresponds not to a matrix group, but to the Symmetric Space $\mathrm{U}(N)/\mathrm{O}(N)$ (see [24], [11]):

$$P_{N,1}(\theta_1,\dots\theta_N) = \mathrm{const}_{N,1}\prod_{1\le k<j\le N}|\exp(i\theta_k)-\exp(i\theta_j)| \tag{96}$$

is the density of the eigenvalue distribution of MM^t, where $M\in\mathrm{U}(N)/\mathrm{O}(N)$.

It is generally assumed, although not proved rigorously, that the short-range correlations between eigenvalues of quantum systems, whose classical analogues are strongly chaotic (geodesic flows on the surfaces with negative curvature, Sinai billiards, Bunimovich stadiums) exhibit C.O.E. statistics ([6], [5], [15]). The point-correlation functions for the C.O.E. are calculated in [11]. They are again of determinantal nature, and moreover they are now the determinants of some $n\times n$ quaternion matrices. We will state these results in a more precise way. Consider quaternions as 2×2 matrices with complex coefficients

$$q=\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

The quaternion units are

$$X=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},\quad Y=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\quad Z=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$$

and

$$q=\frac{a+d}{2}\mathrm{Id}+i\left(\frac{d-a}{2}\right)Z-i\left(\frac{b+c}{2}\right)X+\frac{c-b}{2}Y.$$

Cutting a $2N \times 2N$ matrix $A(M)$ with real or complex coefficients into 2×2 blocks, we can view it as a $N \times N$ quaternion matrix M. The quaternion determinant of M is defined as

$$(97) \qquad QDet M = \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{1}^{l} 1/2 \mathrm{Tr}\left(M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}\right)$$

where the sum is over all permutations, and the factors in the product correspond to the decomposition of σ into cycles. If M is self-dual, i.e.,

$$M_{ji} = (\mathrm{Tr} M_{ij})\mathrm{Id} - M_{ij}, \quad i, j = 1, \dots N,$$

then after the agreement on the order of factors in (97), the summation over all cyclic permutations will give a scalar matrix, and we can omit taking the trace in the formula. Moreover, in this case $(QDet\ M)^2 = \det(A(M))$ (see [12], [23], [24]). Define the function $\Upsilon_N(r)$ as a quaternion

$$\Upsilon_N(r) = \begin{pmatrix} S_N(r), & DS_N(r) \\ JS_N(r), & S_N(r) \end{pmatrix},$$

where

$$(98) \qquad S_N(r) = \sum_{1/2-N/2}^{-1/2+N/2} \exp(ipr) = \frac{\sin(Nr/2)}{\sin(r/2)},$$

$$(99) \qquad D_N(r) = \frac{2\pi}{N}(d/dr) S_N(r) = \frac{2\pi}{N} \sum_{1/2-N/2}^{-1/2+N/2} ip \exp(ipr),$$

$$(100) \qquad JS_N(r) = -\frac{N}{\pi} \sum_{1/2+N/2}^{\infty} l^{-1} \sin(lr).$$

Then n-point correlation functions for the Circular Orthogonal Ensemble are

$$(101) \qquad \rho_n^{(N)}(x_1, \dots x_n) = (2\pi)^{-n} QDet\left(\sigma_N(x_i - x_j)\right)_{i,j=1,\dots n}.$$

We immediately see that in complete analogy with the C.U.E. case (formula (15))

$$(102) \quad r_n^{(N)}(x_1, \dots x_n) = (-1)^{n-1}(2\pi)^{-n} \sum_{\sigma} \Upsilon(x_2 - x_1)\Upsilon(x_3 - x_2) \dots \Upsilon(x_1 - x_n)$$

are the corresponding Ursell functions. Formulas (23), (24), (27), (42), (46) for the correlation functions and (50) for the Ursell functions are still valid, and so are all other combinatorial aspects of the proofs. The main analytical difficulty is that we are not able any longer to claim

$$(2\pi/N)^n \rho_n^{(N)}(x_1, \dots x_n) \le 1$$

since

$$A(M) = \begin{pmatrix} S_N(x_i - x_j), & DS_N(x_i - x_j) \\ JS_N(x_i - x_j), & S_N(x_i - x_j) \end{pmatrix}_{i,j=1,\dots n}$$

is not a positive-definite matrix. More than that, I do not know how to show that

$$(2\pi/N)^n \rho_n^{(N)}(x_1, \dots x_n) \leq C^n \tag{103}$$

where $C > 1$ is arbitrary constant. However, for the purposes of proving Theorems 1.1 and 1.2, the following simpler estimate is sufficient:

LEMMA 6.1.

$$0 \leq (2\pi/N)^n \rho_n^{(N)}(x_1, \dots x_n) \leq (Cn)^{n/2}$$

where $C = 200$.

Proof. Since $M = (\sigma_N(x_i - x_j))_{i,j=1,\dots n}$ is a self-dual matrix,

$$\left((2\pi/N)^n \rho_n^{(N)}(x_1, \dots x_N)\right)^2 = (Q\mathrm{Det}\left(\frac{1}{N}M)\right)^2 = \det \frac{1}{N}A(M).$$

The elements of $A(M)/N$ are uniformly bounded by some constant (10 is enough),

$$|S_N(r)/N| < 10, \quad |DS_N(r)/N| < 10, \quad |JS_N(r)/N| < 10$$

and

$$\mathrm{Tr}\left(\frac{1}{N}A(M) \cdot \frac{1}{N}A(M)^t \leq 10^2(2n)^2\right),$$

which implies

$$\begin{aligned} \det \frac{1}{N}A(M) \cdot \frac{1}{N}A(M)^t &\leq (10^2 2n)^{2n} \\ \det \frac{1}{N}A(M) &\leq (200n)^n \\ (2\pi/N)^n \rho_n^{(N)}(x_1, \dots x_n) &\leq (200n)^{n/2}. \end{aligned}$$

□

In the rescaled coordinates $y_i = (N/2\pi)x_i, y_i \in [0, N], i = 1, \dots N$, the elements of the 2×2 matrix $\Upsilon_N(2\pi y/N)$ decay at infinity as $1/y$:

$$\begin{aligned} |\frac{1}{N}S_N(2\pi y/N)| &< \mathrm{const}(|y| + 1) \\ |\frac{1}{N}DS_N(2\pi y/N)| &< \mathrm{const}(|y| + 1) \\ |\frac{1}{N}JS_N(2\pi y/N)| &< \mathrm{const}(|y| + 1). \end{aligned}$$

Using these inequalities and the one from Lemma 6.1 we can repeat step by step all arguments in the proofs of Theorems 1.1 and 1.2. The correlation function of the limiting Gaussian process $\xi(s)$ in the case of C.O.E. is different from the case of C.U.E. In particular

$$\mathrm{Var}\xi(s) = \lim_{N\to\infty} \mathrm{Var}\eta_N(s)/N \text{ is } \frac{\pi^2}{12}s^2 + O(s^3) \text{ as } s \to 0.$$

However, it is reasonable to conjecture that after choice of the natural time parameter $t = F(s)$ the distribution of the limiting Gaussian processes in the C.U.E. and C.O.E. cases should coincide.

Remark. The proof of Corollary 1.5 requires an estimate of the type (103), which we are not ready to claim at this time.

7. Generalizations and concluding remarks

A) Our methods allow direct generalization to the case of k-level spacings distribution. Namely, one can define a random variable $\eta_N(l,s)$ as a number of eigenvalues that have exactly l neighbors within the distance $2\pi s/N$ to the right. (The distribution of $\eta_N(0,s)$ has been studied in our paper.) It is straightforward to prove similar results for the k-dimensional random process

$$\Big(\big(\eta_N(0,s) - \mathbb{E}\eta_N(0,s)\big)/N^{1/2}, \dots \big(\eta_N(k-1,s) - \mathbb{E}\eta_N(k-1,s)\big)/N^{1/2}\Big)$$

which in particular would tell us about the global k-level spacings distribution, since the number of k-level spacings greater than $2\pi s/N$ equals

$$\sum_{l=0}^{k-1} \eta_N(l,s).$$

One can also count spacings with the help of smooth functions $G : R^k \to R$ with compact support. If $\tau_j = (\theta_{j+1} - \theta_j)N/(2\pi)$ are normalized spacings, then the central limit theorem holds for the statistics

$$\mathcal{G} = \sum_{j=1}^{N} G(\tau_j, \tau_{j+1}, \dots \tau_{j+k-1}) \quad \text{as well.}$$

B) All our results are valid for the general random field defined by n-point correlation functions (11)

$$\rho_n(x_1, \dots x_n) = \det\left(\hat{v}(x_i - x_j)\right)_{i,j=1,\dots n},$$

provided $\hat{v}(x)$ decays at infinity as $O(1/x)$. In particular, similar results should hold for the Gaussian Orthogonal and Unitary Ensembles (see [24] for the definition of the ensembles) in the bulk of the spectrum.

C) In the case of the Circular Symplectic Ensemble ($\beta = 4$), n-point correlation functions are again given by the quaternion-determinants (101) with

$$\Upsilon_N(r) = \frac{1}{2}\begin{pmatrix} S_{2N}(r), & DS_{2N}(r) \\ IS_{2N}(r), & S_{2N}(r) \end{pmatrix},$$

where S_{2N}, DS_{2N} are defined as in Section 6 and

$$IS_{2N}(r) = (N/\pi)\sum_{1/2-N}^{-1/2+N} p^{-1}\sin(pr) = JS_{2N}(r) + \varepsilon_{2N}(r),$$

where

$$\varepsilon_{2N}(r) = \begin{cases} (-1)^m N, & 2\pi m < r < 2\pi(m+1), m = 0, \pm 1, \pm 2, \dots \\ 0, & r = 2\pi m \end{cases}$$

([11], [24]). One can see that in the rescaled coordinates $y_i = (N/2\pi)x_i, i = 1, \dots N$, the quaternion component

$$IS(y) = \lim_{N\to\infty}\frac{1}{2N} IS_{2N}(2\pi y/N) = \operatorname{sgn}(y)\cdot\int_0^{2|y|}\frac{\sin(\pi t)}{\pi t}dt$$

has nonzero limits at $\pm\infty$, which in particular implies that the limiting two-point Ursell function

$$r_2(0,x) = -\left(\frac{\sin(2\pi x)}{2\pi x}\right)^2 + \frac{1}{2}\int_0^{2x}\frac{\sin(\pi t)}{\pi t}dt\cdot(d/dx)\left(\frac{\sin(2\pi x)}{2\pi x}\right)$$

decays at infinity as $1/x$, not $1/x^2$ (which is the case for C.U.E. and C.O.E.). In general, more subtle arguments are required to prove that k-point Ursell functions decay fast enough off the diagonals $x_i = x_j, i, j = 1, \dots N$, to satisfy the conditions of Theorem 2.1. We will return to this problem somewhere else.

PRINCETON UNIVERSITY, PRINCETON, NJ
Current address: CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA
E-mail address: sashas@cco.caltech.edu

REFERENCES

[1] T. H. BAKER and P. J. FORRESTER, Finite N fluctuation formulas for random matrices, J. Stat. Phys. **88** (1997), 1371–1385.

[2] E. BASOR, Distribution functions for random variables for ensembles of positive Hermitian matrices, preprint.

[3] P. BILLINGSLEY, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.

[4] P. BLEHER and A. ITS, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, preprint.

[5] O. Bohigas, Random matrix theories and chaotic dynamics, in: *Chaos and Quantum Physics*, eds. M. J. Giannoni, A. Voros, and J. Zinn-Justin, North Holland, Amsterdam, 1991.

[6] O. Bohigas, M.J. Giannoni, and C. Schmidt, Characterization of chaotic quantum spectra and universality of level fluctuation laws, Phys. Rev. Lett. **52** (1984), 1–4.

[7] É. Brézin and A. Zee, Universality of the correlations between eigenvalues of large random matrices, Nuclear Phys. B **402** (1993), 613–627.

[8] O. Costin and J. Lebowitz, Gaussian fluctuations in random matrices, Phys. Rev. Lett. **75** (1995), 69–72.

[9] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, Studies in Appl. Probab., Essays in honour of Lajos Takacs, J. Appl. Probab., Special Vol. **31A** (1994), 49–62.

[10] F. J. Dyson, Statistical theory of the energy levels of complex systems. I,II,III, J. Mathematical Phys. **3** (1962), 140–175.

[11] ———, Correlations between eigenvalues of a random matrix, Comm. Math. Phys. **19** (1970), 235–250.

[12] F. J. Dyson, The Coulomb fluid and the fifth Painlevé transcendent, in: *Chen Ning Yang. A Great Physicist of the Twentieth Century.* eds. C.-S. Liu and S.- T. Yau, International Press, Cambridge, MA, 1995.

[13] W. Feller, *An Introduction to Probability Theory and its Applications, Vol.* II, John Wiley & Sons, New York, 1966.

[14] M. M. Fogler and B. I. Shklovskii, Probability of an eigenvalue number of fluctuation in an interval of a random matrix spectrum, Phys. Rev. Lett. **74** (1995), 3312–3315.

[15] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer-Verlag, New York, 1990.

[16] K. Johansson, On Szegö's asymptotic formula for Toeplitz determinants and generalizations, Bull. Sci. Math. **112** (1988), 257–304.

[17] ———, On random matrices from the compact classical groups, Ann. of Math. **145** (1997), 519–545.

[18] ———, On fluctuations of eigenvalues of random Hermitian matrices, Duke Math. J. **91** (1998), 151–204.

[19] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloq. Series, to appear, 1998.

[20] J. L. Lebowitz and J. K. Percus, Long-range correlations in a closed system with applications to nonuniform fluids, Phys. Rev. **122** (1961), 1675–1691.

[21] A. Lenard, Correlation functions and the uniqueness of the state in classical statistical mechanics, Comm. Math. Phys. **30** (1973), 35–44.

[22] ———, States of classical statistical mechanical system of infinitely many particles. I, II, Arch. Rational Mech. Anal. **59** (1975), 219–256.

[23] M. L. Mehta, *Matrix Theory*, Editions de Physique, Orsay, France, 1989.

[24] ———, *Random Matrices*, 2^{nd} edition, Academic Press, Boston, 1991.

[25] S. Miller, Cusp forms in $\mathrm{SL}(3, Z)\backslash\mathrm{SL}(3, R)/\mathrm{SO}(3, R)$, Ph.D. Thesis, Princeton University, 1997.

[26] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Proc. Sympos. Pure Math.* **XXIV**, A. M. S., Providence (1973), 181–193.

[27] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, Math. Comp. **48** (1987), 273–308.

[28] ———, The 10^{20} zero of the Riemann zeta function and 70 million of its neighbors, preprint.

[29] L. Pastur and M. Shcherbina, Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles, J. Statist. Phys. **86** (1997), 109–147.

[30] C. E. Porter, *Statistical Theories of Spectra: Fluctuations*, Academic Press, New York, 1965.

[31] Z. RUDNICK and P. SARNAK, Zeros of principal L-functions and random matrix theory, A celebration of John F. Nash, Jr., Duke Math. J. **81** (1996), 269–322.

[32] D. RUELLE, *Statistical Mechanics: Rigorous Results*, W. A. Benjamin, New York, 1969.

[33] H. SPOHN, Interacting Brownian particles: a study of Dyson's model, in: *Hydrodynamic Behavior and Interacting Particle Systems*, G. Papanicolau ed., Springer-Verlag , New York, 1987.

[34] C. A. TRACY and H. WIDOM, Introduction to random matrices, *Geometric and Quantum Aspects of Integrable Systems*, Lecture Notes in Physics **424**, Springer-Verlag, New York, 1993, 103–130.

[35] H. WEYL, *The Classical Groups: Their Invariants and Representations*, Princeton Univ. Press, Princeton, 1939.

[36] H. WIDOM, The asymptotics of a continuous analogue of orthogonal polynomial, J. Approx. Theory **77** (1994), 51–64.

[37] E. P. WIGNER, On a class of analytic functions from the quantum theory of collisions, Ann. of Math. **53** (1951), 36–67.

[38] E. P. WIGNER, Characteristic vectors of bordered matrices with infinite dimensions: I, Ann. of Math. **62** (1955), 548–564

[39] ———, Characteristic vectors of bordered matrices with infinite dimensions: II, Ann. of Math. **65** (1957), 203–207.

[40] ———, On the distribution of the roots of certain symmetric spaces, Ann. of Math. **67** (1958), 325–327.

(Received September 20, 1996)