MODULES OVER A RING (OR OVER AN ALGEBRA)

DIFFERENTIAL =parity reversing homomorphism d of $\mathbb{Z}_2$- graded module obeying $d^2 = 0$

DIFFERENTIAL MODULES = $\mathbb{Z}_2$- graded modules equipped with a differential

HOMOLOGY = Ker $d/\mathrm{Im}\ d$

QUASIISOMORPHISM =homomorphism of differential modules inducing an isomorphism on homology

FREE RESOLUTION =free differential module, quasiisomorphic to original module

DIFFERENTIAL ALGEBRA =algebra equipped with a parity reversing derivation with zero square

DERIVED CATEGORY

SPACE OF STATES

BRST OPERATOR

SPACE OF STATES IN BRST FORMALISM

PHYSICAL STATES

PHYSICAL EQUIVALENCE

BRST FORMALISM

ALGEBRA OF OPERATORS IN BRST FORMALISM

BRST= Becchi, Rouet, Stora, Tyutin

BATALIN - VILKOVISKY (BV) FORMALISM

classical mechnical system =solution to the classical master equation $\{S,S\}=0$ where $\{,\}$ stands for odd Poisson bracket

Q-manifold = supermanifold equipped with odd vector field Q that determines a differential on the space of functions

classical mechnical system=odd symplectic Q-manifold

quantum mechanical system = solution to quantum master equation $\Delta e^{S/\hbar}=0$

Δ -odd Laplacian

$$\bar{A}=\int_L Ae^{S/\hbar}$$

A-quantum observable, i.e. $\Delta A=0$,L- Lagrangian submanifold,playing the role of gauge condition

A. Connes

space $\Longleftrightarrow$ algebra of functions

compact topological space $\Longleftrightarrow$ continuous functions (Gelfand-Naimark)

smooth manifold $\Longleftrightarrow$ smooth functions

formal n-dimensional manifold $\Longleftrightarrow$ formal power series $\mathbb{C}[[x^1, ..., x^n]]$

formal (n,m)-dimensional supermanifold $\Longleftrightarrow$ $\mathbb{C}[[x^1, ..., x^n]] \otimes \Lambda(\xi^1, \ldots, \xi^m)$

NC space $\Longleftrightarrow$ associative algebra

NC superspace $\Longleftrightarrow$ $\mathbb{Z}_2$-graded associative algebra

supermanifold$\Rightarrow$ supercommutative associative algebra vector field on NC space $\Longleftrightarrow$ derivation of algebra

odd vector field on NC superspace $\Longleftrightarrow$ odd derivation of $\mathbb{Z}_2$-graded algebra

NC Q-space $\Longleftrightarrow$ differential $\mathbb{Z}_2$-graded algebra $\Longleftrightarrow$ $\mathbb{Z}_2$-graded algebra equipped with an odd derivation d obeying $d^2 = 0$

Quasiisomorphism of NC Q-spaces $\Longleftrightarrow$ homomorphism of differential $\mathbb{Z}_2$-graded algebras that generates an isomorphism on homology

L_∞ algebra$\Longleftrightarrow$ formal Q-manifold $\Longleftrightarrow$ a differential on the algebra $\mathbb{C}[[x^1,...,x^n]]\otimes\Lambda(\xi^1,\dots,\xi^m)$

differential Lie algebra $\Rightarrow$ L_∞ algebra

zero locus of vector field $Q\Longleftrightarrow$solutions to Maurer-Cartan equations for L_∞ algebra (=usual MC equations in the case of differential Lie algebra)

A_∞ algebra$\Longleftrightarrow$ formal noncommutative Q-manifold $\Longleftrightarrow$ a differential on completion

$$\mathbb{C} << x^1,...,x^n,\xi^1,\dots,\xi^m >>$$

of free non-unital algebra $\mathbb{C} < x^1,...,x^n,\xi^1,\dots,\xi^m >$ generated by $\mathbb{Z}_2$-graded vector space with even coordinates $x^1,...,x^n$ and odd coordinates $\xi^1,\dots,\xi^m$

differential associative algebra $\Rightarrow$ A_∞ algebra

A_∞ algebra A $\Rightarrow$ L_∞ algebra $L(A)$

MAXIMALLY SUPERSYMMETRIC GAUGE THEORIES

10D SUSY YM theory

Its reductions to four, one and zero dimensions: N=4 four-dimensional SUSY YM theory ,BFSS model IKKT model

SUSY YM theory on NC torus

SUSY YM theory on NC torus can be obtained from BFSS or IKKT model by means of compactification (Connes- Douglas -Schw)

Morita equivalence of NC tori $\Rightarrow$ T-duality (Rieffel-Schw,Schw)

Noncommutative instantons (Nekrasov-Schw)

BPS states in NC SUSY YM (Konechny-Schw)

Background independence (Pioline-Schw, Seiberg-Witten)

Algebraic formulation of SUSY YM and more general YM theories in BV formalism in terms of differential associative algebras and A_∞-algebras (Movshev-Sch)

Classical system in Batalin-Vilkovisky (BV) formalism $\Longleftrightarrow$ solution to classical master eqution $\{S,S\}=0$ on odd symplectic manifold $\Longleftrightarrow$ odd symplectic Q-manifold $\Rightarrow$ L_∞ algebra with odd inner product

A_∞-algebra $A \Rightarrow A_\infty$-algebra $A \otimes Mat_n \Rightarrow L_\infty$ algebra $L(A \otimes Mat_n)$

A_∞-algebra A with inner product $\Rightarrow L_\infty$ algebra $L(A \otimes Mat_n)$ with inner product $\Rightarrow$ action functional in BV-formalism

Gauge theories can be obtained this way from appropriate A_∞-algebras

SUSY YM theory can be obtained from differential associative algebra with inner product $\Rightarrow$ BV-action functional can be written in Chern-Simons form

Take

$M = \mathcal{P}/\mathcal{U}(5) \times$(one-dim odd subgroup)

$\mathcal{P}$-super Poincare group,$\mathcal{U}(5)$-parabolic subgroup corresponding to $U(5)$

Differential algebra Ω of $(0,k)$-forms on M with differential $\bar{\partial}$

Odd inner product (odd trace)

Generalized B-model

Similarity with Witten's treatment of N=4 SUSY YM (isotwistors versus twistors)

Infinitesimal deformations of A_∞ algebra $A \Longleftrightarrow$ Hochschild cohomology $HH(A, A)$

Infinitesimal deformations of A_∞ algebra with invariant inner product $\Longleftrightarrow$ cyclic cohomology (Penkava-Schw)

Equivariant generalizations of these statements can be used to analyze SUSY deformations of SUSY YM

TOPOLOGICAL QUANTUM FIELD THEORIES

Action functional does not depend on metric (Sch 1978, 1987)

Chern-Simons action functional $\Rightarrow$ Jones polynomial

Witten type TQFT -the dependence of metric is BRST trivial (1988)

TWO-DIMENSIONAL TOPOLOGICAL THEORY COUPLED TO GRAVITY $\Rightarrow$ STRING THEORY

Topological sigma-models

A-model counts (pseudo)holomorphic curves in symplectic manifolds

B-model is related to variation of complex structure on Calabi-Yau manifold

MIRROR SYMMETRY

ω is a form on Λ and a function on the manifold

If $\xi = 0$ and Γ is a cycle on Λ we can integrate ω_n over Γ to obtain a function on the manifold. Integrating this function over Lagrangian submanifold L we obtain a number depending only on homology classes of Γ and L

FAMILIES OF GAUGE CONDITIONS=FAMILIES OF EQUIVALENT ACTION FUNCTIONALS

M odd symplectic manifold

$$\lambda \in \Lambda$$

$$\Delta S_\lambda = 0, \{S_\lambda, S_\lambda\} = 0$$

$$\delta_V S_\lambda = \{B(V), S\}, \Delta B(V) = 0$$

$$\xi(V_1, V_2) = B[V_1, V_2] - \hat{V}_1 B(V_2) + \hat{V}_2 B(V_1) + \{B(V_1), B($$

$\xi(V_1, V_2)$ is a closed two-form on Λ and a function on the manifold

$$\omega = \sum \omega_n = \sum B(V_1) \ldots B(V_n) e^S$$

$$\Delta\omega = (d + \xi)\omega$$