

# Is there a topological theory of flux vacua ?

Michael R. Douglas  
Rutgers, I.H.E.S. and Caltech

For Albert Schwarz, UC Davis, May 2004

## Abstract

Based on hep-th/0303194, hep-th/0307049 with Su-  
jay Ashok, math.CV/0402326 with Bernard Shiffman and  
Steve Zelditch (Johns Hopkins), hep-th/0404116 and to ap-  
pear with Frederik Denef, and hep-th/0404257 with Frederik  
Denef and Bogdan Florea.

[Home Page](#)[Title Page](#)[Page 1 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

# 1. Introduction

Recent work on superstring compactification has demonstrated the importance of non-zero gauge field strengths (or **flux**) for problems such as stabilizing moduli and obtaining a small cosmological constant.

In this talk, we will discuss a new class of counting problems, counting flux vacua, which admit a very simple solution in terms of topological invariants of Calabi-Yau moduli space.

*Introduction*

*Vacua in . . .*

*Flux vacua*

*Distributions of . . .*

*Kähler moduli . . .*

*Conclusions*

*Home Page*

*Title Page*



*Page 2 of 31*

*Go Back*

*Full Screen*

*Close*

*Quit*

# 1. Introduction

Recent work on superstring compactification has demonstrated the importance of non-zero gauge field strengths (or **flux**) for problems such as stabilizing moduli and obtaining a small cosmological constant.

In this talk, we will discuss a new class of counting problems, counting flux vacua, which admit a very simple solution in terms of topological invariants of Calabi-Yau moduli space.

What we would like to know is,

Is there a topological field theory formulation of the problem of counting flux vacua ?

If so, we think this might be the first of a new class of topological theories.

[Home Page](#)[Title Page](#)[◀◀](#)[▶▶](#)[◀](#)[▶](#)[Page 2 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

# 1. Introduction

Recent work on superstring compactification has demonstrated the importance of non-zero gauge field strengths (or **flux**) for problems such as stabilizing moduli and obtaining a small cosmological constant.

In this talk, we will discuss a new class of counting problems, counting flux vacua, which admit a very simple solution in terms of topological invariants of Calabi-Yau moduli space.

What we would like to know is,

Is there a topological field theory formulation of the problem of counting flux vacua ?

If so, we think this might be the first of a new class of topological theories.

We will then give an overview of recent results on the distribution of flux vacua.

[Home Page](#)[Title Page](#)[⏪](#) [⏩](#)[◀](#) [▶](#)[Page 2 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

## 2. Vacua in supergravity

We begin by reviewing the definition of supersymmetric vacua in theories with  $d = 4$ ,  $\mathcal{N} = 1$  supersymmetry. There are two main classes of theory: globally supersymmetric, and locally supersymmetric (supergravity theory).

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)

◀◀

▶▶

◀

▶

Page 3 of 31

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 2. Vacua in supergravity

We begin by reviewing the definition of supersymmetric vacua in theories with  $d = 4$ ,  $\mathcal{N} = 1$  supersymmetry. There are two main classes of theory: globally supersymmetric, and locally supersymmetric (supergravity theory).

In both cases, the basic data specifying a supersymmetric theory  $T$  is a triple  $(\mathcal{C}, K, W)$ , where

- $\mathcal{C}$  is the “configuration space,” a complex Kähler manifold of dimension  $d$ . Let local complex coordinates be  $z^i$ .
- $K$  is a Kähler potential, determining the Kähler metric on  $\mathcal{C}$ . Let  $\omega = \frac{i}{2} \partial \bar{\partial} K$  be the Kähler form.
- $W$  is the superpotential, locally an analytic function on  $\mathcal{C}$ .

## 2. Vacua in supergravity

We begin by reviewing the definition of supersymmetric vacua in theories with  $d = 4$ ,  $\mathcal{N} = 1$  supersymmetry. There are two main classes of theory: globally supersymmetric, and locally supersymmetric (supergravity theory).

In both cases, the basic data specifying a supersymmetric theory  $T$  is a triple  $(\mathcal{C}, K, W)$ , where

- $\mathcal{C}$  is the “configuration space,” a complex Kähler manifold of dimension  $d$ . Let local complex coordinates be  $z^i$ .
- $K$  is a Kähler potential, determining the Kähler metric on  $\mathcal{C}$ . Let  $\omega = \frac{i}{2} \partial \bar{\partial} K$  be the Kähler form.
- $W$  is the superpotential, locally an analytic function on  $\mathcal{C}$ .

The difference between global and local supersymmetry comes with the precise definition of the superpotential  $W$ , and of supersymmetric vacua.

In global supersymmetry,  $W$  is a holomorphic function on  $\mathcal{C}$  (possibly with singularities), and a supersymmetric vacuum is a critical point

$$0 = \frac{\partial}{\partial z^i} W(z).$$

In local supersymmetry,  $W$  is a holomorphic section of a line bundle  $\mathcal{L}$  with curvature

$$c_1(\mathcal{L}) = -\frac{1}{\pi}\omega = -\frac{i}{2\pi}\partial\bar{\partial}K.$$

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)

[◀](#)

[▶](#)

[◀](#)

[▶](#)

Page 4 of 31

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

In local supersymmetry,  $W$  is a holomorphic section of a line bundle  $\mathcal{L}$  with curvature

$$c_1(\mathcal{L}) = -\frac{1}{\pi}\omega = -\frac{i}{2\pi}\partial\bar{\partial}K.$$

It can be defined as follows:

- Choose a local frame  $e_{\mathcal{L}}$ .
- Define a hermitian metric  $h$  by

$$K(z, \bar{z}) = -\log |e_L(z)|_h^2. \quad (1)$$

- The Chern connection  $D$  is a holomorphic connection which preserves the hermitian metric,

$$D(f e_L) = (df - f\partial K) \otimes e_L. \quad (2)$$

Acting on a holomorphic section  $s$ ,  $Ds$  is purely  $(1,0)$ .

We will use the physics notation  $e^K |W|_h^2$  for the norm of the section,  $|W|_h^2$

A supersymmetric vacuum in supergravity is then a **critical point** of the section  $W$ ,

$$D W = 0,$$

or in local coordinates,

$$0 = D_i W = \frac{\partial W}{\partial z^i} + \left( \frac{\partial K}{\partial z^i} \right) W.$$

[Home Page](#)[Title Page](#)[Page 5 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

A supersymmetric vacuum in supergravity is then a **critical point** of the section  $W$ ,

$$D W = 0,$$

or in local coordinates,

$$0 = D_i W = \frac{\partial W}{\partial z^i} + \left( \frac{\partial K}{\partial z^i} \right) W.$$

There are two types of critical point, Where  $W \neq 0$ , these are ordinary critical points of the function

$$\hat{\Lambda} = |W(z)|_h^2.$$

Thus, our problem has a close relation to the Morse theory of the function  $\hat{\Lambda}$ .

However, it is not the same, because every point where  $W = 0$  is a critical point of  $\hat{\Lambda}$ . In general these are not critical points of  $W$ ;  $DW = 0$  reduces to  $\partial W = 0$  and these are singular points of the hypersurface  $W = 0$ .

Physically, both types of critical points are supersymmetric vacua. The vacuum energy or cosmological constant is  $-3\hat{\Lambda}$ , so  $W = 0$  are Minkowski vacua, and  $W \neq 0$  are AdS vacua.

[Home Page](#)
[Title Page](#)
[Page 5 of 31](#)
[Go Back](#)
[Full Screen](#)
[Close](#)
[Quit](#)

### 3. Flux vacua

The simplest example of an  $(C, K, W)$  which arises in compactification of string theory is the following.

We take  $\mathcal{C} = \mathcal{H}$  the upper half plane with complex coordinate  $\tau$ , and  $K = -\log \text{Im } \tau$ , the Kähler potential for the constant negative curvature metric.

And we take

$$W = A\tau + B$$

for complex constants  $A, B$ .

### 3. Flux vacua

The simplest example of an  $(C, K, W)$  which arises in compactification of string theory is the following.

We take  $\mathcal{C} = \mathcal{H}$  the upper half plane with complex coordinate  $\tau$ , and  $K = -\log \text{Im } \tau$ , the Kähler potential for the constant negative curvature metric.

And we take

$$W = A\tau + B$$

for complex constants  $A, B$ .

Now it is easy to solve the equation  $DW = 0$ :

$$\begin{aligned} DW &= \frac{\partial W}{\partial \tau} - \frac{1}{\tau - \bar{\tau}} W \\ &= \frac{-A\bar{\tau} - B}{\tau - \bar{\tau}} \end{aligned}$$

so  $DW = 0$  at

$$\bar{\tau} = -\frac{B}{A}$$

where  $\bar{\tau}$  is the complex conjugate.

[Home Page](#)
[Title Page](#)
[Page 6 of 31](#)
[Go Back](#)
[Full Screen](#)
[Close](#)
[Quit](#)

This supergravity problem comes from IIB string theory compactified on a Calabi-Yau  $M$  with a Ricci-flat metric  $g$ . The space of Ricci-flat metrics on  $M$  has parameters (moduli), each of which could become a massless field in four-dimensional space-time, with at least gravitational strength couplings. Such fields are in contradiction with experiment.

By Yau's theorem, the moduli parameterize a choice of complex structure on  $M$ , and a choice of Kähler class. Finally, there is a massless field  $\tau$  called "dilaton-axion," which controls the fundamental string coupling constant. All of these fields need to be fixed, both to avoid contradictions with experiment and to predict the coupling constants of nature.

[Home Page](#)[Title Page](#)[Page 7 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

Here is one way to do it. The IIB theory has two two-form potentials  $B^{(2)}$  and  $C^{(2)}$ , which minimally couple to the fundamental string and the D-string respectively. The action in ten dimensions includes the terms

$$\int d^{10}x \, dC \wedge *dC + |\tau|^2 dB \wedge *dB \quad (3)$$

generalizing the Maxwell action from one-form to two-form potentials.

We seek solutions with non-zero field strengths (flux)  $dB$  and  $dC$ . The generalized Maxwell's equations,  $d * dB = d * dC = 0$ , require these to be **harmonic**. Furthermore, they satisfy the analog of the Dirac quantization condition on  $F = dA$ ,

$$\int_{\Sigma^{(3)}} (dB \, dC) = 2\pi \left( \vec{A} \cdot \vec{B} \right); \quad n \in \mathbb{Z}.$$

So there are a discrete set of “flux vacua” labelled by the periods  $(A \ B)$  on every three-cycle. From the action (3), the energy of a flux vacuum depends on  $\tau$ , and in general on the metric on the CY. Thus, minimizing the energy fixes  $\tau$  and the (complex structure) moduli of the Ricci-flat metric.

It can be shown (Gukov, Vafa, Witten; Giddings, Kachru, Polchinski) that supersymmetric minima can be described by supergravity as above, with

$$W = \int_M \Omega^{(3)}(z) \wedge (dC + \tau dB)$$

Consider a rigid CY, i.e. with  $b^{2,1} = 0$  (for example, the orbifold  $T^6/\mathbb{Z}_3$ ). Then, we get the problem we just discussed, with

$$W = A\tau + B; \quad A = a_1 + \Pi a_2; B = b_1 + \Pi b_2$$

and  $K = -\log \text{Im } \tau$ .

Here  $\Pi = \int_{\Sigma_2} \Omega^{(3)} / \int_{\Sigma_1} \Omega^{(3)}$ , a constant determined by CY geometry. More generally, what appears here are vectors of periods depending on the complex structure.

Conjecture: there are **finitely many vacua of string and M theory**, that look even minimally like the real world, in having

- Four macroscopic dimensions.
- No massless scalar fields.
- Cosmological constant significantly below the Planck scale.

This is almost certainly necessary for any hopes of testing or making predictions from string theory. And, from our present understanding of the theory, it is not at all obvious.

So how many flux vacua are there? There will be infinitely many, unless something bounds the allowed values of flux.

[Home Page](#)[Title Page](#)[Page 10 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

In this problem, there is a natural constraint on the flux, coming from the following term in the ten dimensional action,

$$\int d^{10}x C^{(4)} \wedge dB^{(2)} \wedge dC^{(2)}.$$

It produces a “RR tadpole” which must be compensated by certain singularities in the compactification (orientifold 3-planes) for consistency. This leads to the condition

$$L = \int_M dB^{(2)} \wedge dC^{(2)}$$

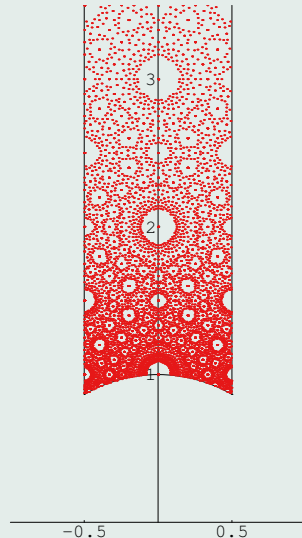
for some fixed number  $L$ .

So, we would like to count solutions of  $DW = 0$ , summed over choices of flux  $dB$  and  $dC$  satisfying this constraint. Furthermore, there is a **duality group**  $SL(2, \mathbb{Z})$  which equates solutions related by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}; \quad (A \ B) \rightarrow (aA + cB \ bA + dB).$$

We would like to count one solution in each orbit of this group. This can be done by restricting  $\tau$  to a fundamental region of the upper half plane.

Here is the resulting set of flux vacua for  $L = 150$  and  $\Pi = i$ :



This graph was obtained by enumerating one solution of  $a_1 b_2 - a_2 b_1 = L$  in each  $SL(2, \mathbb{Z})$  orbit, taking the solution  $\tau = -(b_1 - i b_2)/(a_1 - i a_2)$  and mapping it back to the fundamental region.

The total number of vacua is  $N = 2\sigma(L)$ , where  $\sigma(L)$  is the sum of the divisors of  $L$ . Its large  $L$  asymptotics are  $N \sim \pi^2 L/6$ .

A similar enumeration for a Calabi-Yau with  $n$  complex structure moduli, would produce a similar plot in  $n + 1$  complex dimensions, the distribution of flux vacua. It could (in principle) be mapped into the distribution of **possible values of coupling constants** in a physical theory.

The intricate distribution we just described has some simple properties. For example, one can get exact results for the large  $L$  asymptotics, by computing a continuous distribution  $\rho(z, \tau; L)$ , whose integral over a region  $R$  in moduli space reproduces the asymptotic number of vacua which stabilize moduli in the region  $R$ , for large  $L$ ,

$$\int_R dz d\tau \rho(z, \tau; L) \sim_{L \rightarrow \infty} N(R).$$

For a region of radius  $r$ , the continuous approximation should become good for  $L \gg K/r^2$ .

[Home Page](#)[Title Page](#)[Page 13 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

The intricate distribution we just described has some simple properties. For example, one can get exact results for the large  $L$  asymptotics, by computing a continuous distribution  $\rho(z, \tau; L)$ , whose integral over a region  $R$  in moduli space reproduces the asymptotic number of vacua which stabilize moduli in the region  $R$ , for large  $L$ ,

$$\int_R dz d\tau \rho(z, \tau; L) \sim_{L \rightarrow \infty} N(R).$$

For a region of radius  $r$ , the continuous approximation should become good for  $L \gg K/r^2$ .

Explicit formulas for these densities can be found, in terms of the geometry of the moduli space  $\mathcal{C}$ . The simplest such result computes the [index density](#) of vacua:

$$\rho_I(z, \tau) = \frac{(2\pi L)^{b_3}}{b_3! \pi^{n+1}} \det(-R - \omega \cdot 1)$$

where  $\omega$  is the Kähler form and  $R$  is the matrix of curvature two-forms. This is a lower bound for the number of vacua.

The index density can be defined as

$$\rho_I(z, \tau) = \sum_{\substack{dB, dC \\ L = \int dB \wedge dC}} (-1)^F \delta_{z, \tau}(DW(z, \tau))$$

where

$$(-1)^F = \text{sgn} \det \begin{pmatrix} \bar{\partial}_{\bar{i}} D_j W(z) & \partial_i D_j W(z) \\ \bar{\partial}_{\bar{i}} \bar{D}_{\bar{j}} W^*(z) & \partial_i \bar{D}_{\bar{j}} W^*(z) \end{pmatrix}.$$

This is  $(-1)^{n+1}$  for Minkowski vacua  $W = 0$ , and is the Morse sign for the function  $e^K |W|^2$  for  $W \neq 0$ .

The index is invariant under small variations of the data  $K$  and  $W$  and is thus the obvious candidate for a topological field theory computation. Indeed, if  $\mathcal{C}$  had been a compact manifold, it would be invariant under arbitrary variations,

$$[c_{n+1}(\Omega\mathcal{C} \otimes \mathcal{L})] = \frac{1}{\pi^{n+1}} \int_{\mathcal{C}} \det(-R - \omega \cdot 1).$$

So why not compute it with a topological field theory, say a variation on the supersymmetric quantum mechanics which could be used to count solutions of  $V'(z) = 0$  on a compact manifold,

$$\sum_{V'(z)=0} (-1)^F = \text{Tr} (-1)^F = \chi(\mathcal{C})$$

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)



Page 15 of 31

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

So why not compute it with a topological field theory, say a variation on the supersymmetric quantum mechanics which could be used to count solutions of  $V'(z) = 0$  on a compact manifold,

$$\sum_{V'(z)=0} (-1)^F = \text{Tr} (-1)^F = \chi(\mathcal{C})$$

The first problem we run into, is that the number of solutions of  $DW = 0$  in fact depends on  $W$ . This is because we only count solutions which sit in the fundamental region of  $SL(2, \mathbb{Z})$  (or more general Calabi-Yau monodromy group).

Only the total number of flux vacua, obtained by summing over fluxes, is “topological.” One needs to cook up a field theory which makes the constrained sum over fluxes we described.

One might think this could be done by topologically twisting  $N = 1$  supergravity, but this appears problematic.

Anyways, the index density can be integrated over a fundamental region of the moduli space to estimate the total number of flux vacua. For example, for  $T^6$  (with symmetrized period matrix),  $K = b_3 = 20$ , and

$$I = \int \rho^I = \frac{7 \cdot (2\pi L)^{20}}{4 \cdot 181440 \cdot 12 \cdot 20!} \sim 4 \cdot 10^{21} \quad \text{for } L = 32.$$

Anyways, the index density can be integrated over a fundamental region of the moduli space to estimate the total number of flux vacua. For example, for  $T^6$  (with symmetrized period matrix),  $K = b_3 = 20$ , and

$$I = \int \rho^I = \frac{7 \cdot (2\pi L)^{20}}{4 \cdot 181440 \cdot 12 \cdot 20!} \sim 4 \cdot 10^{21} \quad \text{for } L = 32.$$

More examples: the “mirror quintic” ([Greene and Plesser](#); [Candelas et al](#)) with a one parameter moduli space  $\mathcal{M}_c(\tilde{Q})$ . The integral

$$\frac{1}{\pi^2} \int_c \det(-R - \omega) = \frac{1}{12} \chi(\mathcal{M}_c(\tilde{Q})) = \frac{1}{60}.$$

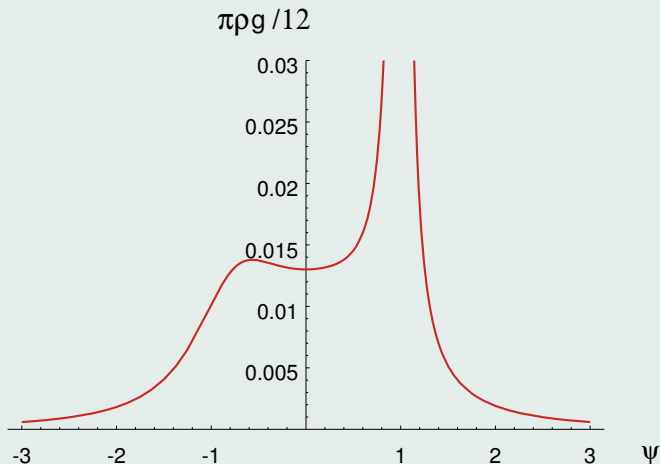
And, the quintic, with its 101 parameter moduli space ([Z. Lu](#)),

$$\int_{\mathcal{M}_c(Q)} \frac{\omega^{101}}{101!} = 5^{-24} \frac{1}{|\Gamma|}$$

where  $\Gamma$  is a residual discrete symmetry group.

## 4. Distributions of flux vacua

Let us look at the details of the distribution of flux vacua on the mirror quintic ( $K = 4$  and  $n = 1$ ), as a function of complex structure modulus:



Note the divergence at  $\psi = 1$ . This is the conifold point, with a dual gauge theory interpretation. It arises because the curvature  $R \sim \partial\bar{\partial} \log \log |\psi - 1|^2$  diverges there. The divergence is integrable, but a finite fraction of all the flux vacua sit near it.

In a bit more detail, let  $v$  be a modulus near the conifold point  $v = 0$ . The metric near  $v = 0$  is

$$g_{v\bar{v}} \approx c \ln \frac{\mu^2}{|v|^2}, \quad (4)$$

and the third derivative of the prepotential or “Yukawa coupling” is

$$\mathcal{F} = \partial^3 \mathcal{F}(v) = g_{v\bar{v}}^{-3/2} e^K \left( i \int_A \hat{\Omega} \partial_v^3 \int_B \hat{\Omega} + \text{anal.} \right) \approx i \left( c \ln \frac{\mu^2}{|v|^2} \right)^{-3/2} \frac{c}{v}, \quad (5)$$

so  $\mathcal{F} \rightarrow \infty$  when  $v \rightarrow 0$ . The same is true for  $\rho \approx |\mathcal{F}|^2/\pi^2$ . However, the density integrated over the fundamental  $\tau$ -domain and  $|z| < R$  remains finite. For small  $R$ :

$$\int d^2\tau g_{\tau\bar{\tau}} \int d^2v g_{v\bar{v}} \rho \approx \frac{1}{12 \ln \frac{\mu^2}{R^2}}. \quad (6)$$

The number of susy vacua with  $L \leq L_*$  and  $|v| \leq R$  is

$$\mathcal{N}_{vac} = \frac{\pi^4 L_*^4}{18 \ln \frac{\mu^2}{R^2}}. \quad (7)$$

For example, with  $L_* = 100$  and  $\mu = 1$ , there are about one million susy vacua with  $|v| < 10^{-100}$  (about 0.1%).

In a bit more detail, let  $v$  be a modulus near the conifold point  $v = 0$ . The metric near  $v = 0$  is

$$g_{v\bar{v}} \approx c \ln \frac{\mu^2}{|v|^2}, \quad (4)$$

and the third derivative of the prepotential or “Yukawa coupling” is

$$\mathcal{F} = \partial^3 \mathcal{F}(v) = g_{v\bar{v}}^{-3/2} e^K \left( i \int_A \hat{\Omega} \partial_v^3 \int_B \hat{\Omega} + \text{anal.} \right) \approx i \left( c \ln \frac{\mu^2}{|v|^2} \right)^{-3/2} \frac{c}{v}, \quad (5)$$

so  $\mathcal{F} \rightarrow \infty$  when  $v \rightarrow 0$ . The same is true for  $\rho \approx |\mathcal{F}|^2/\pi^2$ . However, the density integrated over the fundamental  $\tau$ -domain and  $|z| < R$  remains finite. For small  $R$ :

$$\int d^2\tau g_{\tau\bar{\tau}} \int d^2v g_{v\bar{v}} \rho \approx \frac{1}{12 \ln \frac{\mu^2}{R^2}}. \quad (6)$$

The number of susy vacua with  $L \leq L_*$  and  $|v| \leq R$  is

$$\mathcal{N}_{vac} = \frac{\pi^4 L_*^4}{18 \ln \frac{\mu^2}{R^2}}. \quad (7)$$

For example, with  $L_* = 100$  and  $\mu = 1$ , there are about one million susy vacua with  $|v| < 10^{-100}$  (about 0.1%).

Vacua close to conifold degenerations are interesting for model building, as they provide a natural mechanism for generating large scale hierarchies (Randall and Sundrum, etc.). They may also enable controlled constructions of de Sitter vacua by adding anti-D3 branes, as proposed by KKLT. However, for the latter it is also necessary that the mass matrix at the critical point is positive.

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)



Page 19 of 31

[Go Back](#)

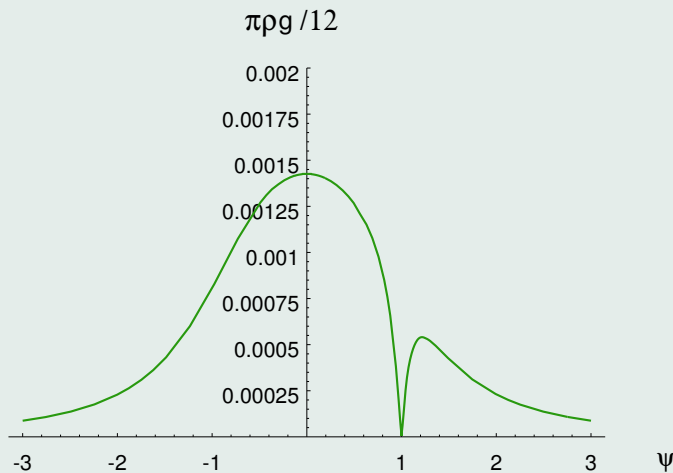
[Full Screen](#)

[Close](#)

[Quit](#)

Vacua close to conifold degenerations are interesting for model building, as they provide a natural mechanism for generating large scale hierarchies (Randall and Sundrum, etc.). They may also enable controlled constructions of de Sitter vacua by adding anti-D3 branes, as proposed by KKLT. However, for the latter it is also necessary that the mass matrix at the critical point is positive.

The distribution of tachyon-free D breaking vacua is



In fact, most D breaking vacua near the conifold point have tachyons (for one modulus CY's), so we get suppression, not enhancement.

This is not hard to understand: assuming  $DW = 0$ , the bosonic mass matrix is (as in supersymmetric AdS),

$$M = H^2 - 3e^{K/2}|W|H, \quad H = 2d^2e^{K/2}|W|. \quad (8)$$

This means that eigenvalues of  $H$  (the fermion mass matrix) between 0 and  $3e^{K/2}|W|$  lead to tachyons. In general, small  $|W|$  makes tachyonic moduli unlikely.

In fact, most D breaking vacua near the conifold point have tachyons (for one modulus CY's), so we get suppression, not enhancement.

This is not hard to understand: assuming  $DW = 0$ , the bosonic mass matrix is (as in supersymmetric AdS),

$$M = H^2 - 3e^{K/2}|W|H, \quad H = 2d^2e^{K/2}|W|. \quad (8)$$

This means that eigenvalues of  $H$  (the fermion mass matrix) between 0 and  $3e^{K/2}|W|$  lead to tachyons. In general, small  $|W|$  makes tachyonic moduli unlikely.

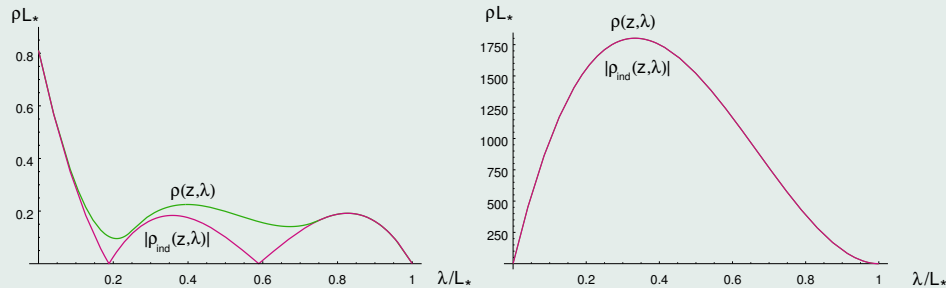
However, in a one modulus model, the matrix  $H$  takes the form

$$\Delta H = \begin{pmatrix} 0 & S \\ \bar{S} & 0 \end{pmatrix}, \quad S = \frac{\bar{W}}{|W|} \begin{pmatrix} 0 & Z \\ Z & \mathcal{F}\bar{Z} \end{pmatrix}. \quad (9)$$

( $Z$  is a random parameter). When  $\mathcal{F}$  is large, its eigenvalues are approximately  $e^K|W| \pm \mathcal{F}^{\pm Z}$ , and there is always a small positive eigenvalue.

It remains to be seen whether this is also true in multi-modulus models.

Here is the distribution of cosmological constants, both at generic points (left) and near the conifold point (right). Note that at generic points it is fairly uniform, all the way to the string scale. On the other hand, imposing small c.c. competes with the enhancement of vacua near the conifold point.



The left hand graph compares the total number of vacua (green) with the index (red). The difference measures the number of **Kähler stabilized vacua**, vacua which exist because of the structure of the Kähler potential, not the superpotential.

[Home Page](#)
[Title Page](#)
[◀](#)
[▶](#)
[◀](#)
[▶](#)

Page 21 of 31

[Go Back](#)
[Full Screen](#)
[Close](#)
[Quit](#)

Another simple universal property: In the large complex structure limit,  $R \sim \omega$  and the density of vacua is reasonably well approximated by  $\det \omega$ . This can be computed from the Kähler potential, which is cubic in the LCS limit,

$$K = -\log c_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma$$

where  $y^\alpha = \text{Im } t^\alpha$  is a Kähler modulus, giving

$$\omega \sim \frac{dt d\bar{t}}{y^2}.$$

[Home Page](#)[Title Page](#)[Page 22 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

Another simple universal property: In the large complex structure limit,  $R \sim \omega$  and the density of vacua is reasonably well approximated by  $\det \omega$ . This can be computed from the Kähler potential, which is cubic in the LCS limit,

$$K = -\log c_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma$$

where  $y^\alpha = \text{Im } t^\alpha$  is a Kähler modulus, giving

$$\omega \sim \frac{dt d\bar{t}}{y^2}.$$

Taking the determinant, one finds  $\det \omega \sim y^{-2n}$ , and this leads to a universal falloff in the large complex structure limit,

$$\int_{V>V_0} \rho \sim V_0^{-n/3}$$

where  $V$  is the “mirror volume” (the mirror of the LCS is the large volume limit). For large  $n$ , this is a drastic falloff, and typically there are no vacua in this regime.

This suppression can be understood directly in terms of the mirror IIa derivation of the metric on Calabi-Yau moduli space. In this case it is the metric on Kähler moduli space, which at large volume comes from the metric on the space of metrics,

$$\langle \delta g_{ij}, \delta g_{kl} \rangle = \frac{1}{V} \int_{CY} \sqrt{g} g^{ik} g^{jl} \delta g_{ij} \delta g_{kl}$$

where  $g_{ij}$  is the metric on the CY, and the  $1/V$  factor (which compensates the  $\sqrt{g}$ ) comes from the standard derivation of the kinetic term in KK reduction on CY. Because of the inverse factors of the metric, this falls off with volume as  $V^{-1/3}$ . This factor appears for each modulus, leading to a dramatic effect with many moduli.

This suppression can be understood directly in terms of the mirror IIa derivation of the metric on Calabi-Yau moduli space. In this case it is the metric on Kähler moduli space, which at large volume comes from the metric on the space of metrics,

$$\langle \delta g_{ij}, \delta g_{kl} \rangle = \frac{1}{V} \int_{CY} \sqrt{g} g^{ik} g^{jl} \delta g_{ij} \delta g_{kl}$$

where  $g_{ij}$  is the metric on the CY, and the  $1/V$  factor (which compensates the  $\sqrt{g}$ ) comes from the standard derivation of the kinetic term in KK reduction on CY. Because of the inverse factors of the metric, this falls off with volume as  $V^{-1/3}$ . This factor appears for each modulus, leading to a dramatic effect with many moduli.

A possible physical application of this: we know how to stabilize complex structure moduli using fluxes in IIb. Suppose we can use T-duality to get a corresponding class of models in IIa with stabilized Kähler moduli. Then, the mirror interpretation of this result is the number of vacua which stabilize the [volume of the compact dimensions](#) at a given value.

Ignoring the geometric factor, and writing  $V \sim R^6$ , we find a number of vacua

$$N \sim \frac{(2\pi L)^K R^{-K}}{K!}$$

so large  $K$  disfavors large volume in this case, and the maximum volume one expects is of order

$$V \sim \left( \frac{2\pi L}{K} \right)^6.$$

The parameter  $L = \chi/24$  in F theory compactification on four-folds and can reach values of several thousand. This will provide the upper bound on values of  $V$ . Will it reach the  $\sim 10^{30}$  of the “large extra dimensions” scenario?

## 5. Kähler moduli stabilization

So far we have been discussing distributions of complex structure and dilaton moduli in IIB string theory and F theory. Because of duality, Kähler moduli are probably not fundamentally different, and it is a good guess that they are governed by the same picture.

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 25 of 31

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 5. Kähler moduli stabilization

So far we have been discussing distributions of complex structure and dilaton moduli in IIB string theory and F theory. Because of duality, Kähler moduli are probably not fundamentally different, and it is a good guess that they are governed by the same picture.

However, with the present state of the art, discussing Kähler moduli stabilization in IIB is quite different – there is no direct analog of classical stabilization by fluxes. Rather, one must find stringy nonperturbative effects to do this, and work in a semi-classical limit. Although results obtained this way are limited, they are important for discussing the questions of whether all moduli can be stabilized, and whether there is any fundamental problem with obtaining positive c.c. .

## 5. Kähler moduli stabilization

So far we have been discussing distributions of complex structure and dilaton moduli in IIB string theory and F theory. Because of duality, Kähler moduli are probably not fundamentally different, and it is a good guess that they are governed by the same picture.

However, with the present state of the art, discussing Kähler moduli stabilization in IIB is quite different – there is no direct analog of classical stabilization by fluxes. Rather, one must find stringy nonperturbative effects to do this, and work in a semi-classical limit. Although results obtained this way are limited, they are important for discussing the questions of whether all moduli can be stabilized, and whether there is any fundamental problem with obtaining positive c.c. .

[Kachru](#), [Kallosh](#), [Linde](#) and [Trivedi](#) suggested to look for vacua with all moduli stabilized by F theory and gauge theory nonperturbative contributions to the superpotential.

Recently, with [Denef and Florea](#), we have found the first concrete examples of vacua in which all moduli can be stabilized.

The nonperturbative effects used in these examples are D3-brane instantons. For these to contribute to the superpotential, they must wrap surfaces which lift to fourfold divisors of arithmetic genus one ([Witten](#)):

$$1 = \chi(\mathcal{O}_D) = h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3}.$$

The a.g. one condition is necessary for a D-instanton to have only two zero modes (from  $h^{0,0}$ ). In gauge theory terms, it follows from the fact that adjoint matter suppresses the superpotential.

Recently, with [Denef and Florea](#), we have found the first concrete examples of vacua in which all moduli can be stabilized.

The nonperturbative effects used in these examples are D3-brane instantons. For these to contribute to the superpotential, they must wrap surfaces which lift to fourfold divisors of arithmetic genus one ([Witten](#)):

$$1 = \chi(\mathcal{O}_D) = h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3}.$$

The a.g. one condition is necessary for a D-instanton to have only two zero modes (from  $h^{0,0}$ ). In gauge theory terms, it follows from the fact that adjoint matter suppresses the superpotential.

In the math literature, there is a very general relation between divisors of a.g. one, and contractions of manifolds. This allows proving the following relation in many cases (toric or Fano base):

$$D \cdot \Sigma < 0$$

for  $\Sigma$  an effective curve. This implies that the instanton action for to the divisor  $D$  must have at least one negative coefficient (in some basis),

$$W \sim b_D \exp 2\pi i(c_1 t_1 + \dots - c_n t_n).$$

This implies that [no model with one Kähler modulus](#) can stabilize Kähler moduli (see also [Robbins and Sethi, 0405011](#)).

Home Page

Title Page

Page 26 of 31

Go Back

Full Screen

Close

Quit

Using the very complete study of divisors of a. g. one of [A. Grassi, math.AG/9704008](#), we have found 6 models with toric Fano threefold base which can stabilize all Kähler moduli, and which could be analyzed in complete detail using existing techniques.

The simplest,  $\mathcal{F}_{18}$ , has 89 complex structure moduli. According to the AD counting formula, it should have roughly  $\epsilon \times 10^{307}$  flux vacua with all moduli stabilized, where

$$\epsilon = g_s^2 \times \left. \frac{|W|}{m_s^4} \right|_{max}.$$

Models which stabilize all Kähler moduli are **not** generic, because a.g. one divisors are not. However, they are not uncommon either; there are 29 out of 100 with toric base, and probably many more with  $\mathbb{P}^1$  fibered base. This last class of model should be simpler, in part because these have heterotic duals.

## 6. Conclusions

The value of the statistical approach to string compactification depends on whether distributions of vacua which are simple enough to write down explicitly and work with, can capture interesting features of the real distribution of string/M theory vacua.

We now have explicit results for distributions of flux vacua of many types: supersymmetric, non-supersymmetric, tachyon-free. They display a lot of structure, with suggestive phenomenological implications:

- Enhanced numbers of vacua near conifold points (possibly inconsistent with metastability and small c.c.).
- Correlations with the cosmological constant
- Falloff in numbers at large volume and large complex structure.

Another result: the distribution of the parameter  $-3e^K|W|^2$  in flux vacua is fairly uniform, all the way to the string scale. This means that an arbitrary supersymmetry breaking contribution to the vacuum energy,

$$V = e^K(|DW|^2 - 3|W|^2) + D^2,$$

can be compensated by the negative term, with no preferred scale.

[Home Page](#)[Title Page](#)[Page 29 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

Another result: the distribution of the parameter  $-3e^K|W|^2$  in flux vacua is fairly uniform, all the way to the string scale. This means that an arbitrary supersymmetry breaking contribution to the vacuum energy,

$$V = e^K(|DW|^2 - 3|W|^2) + D^2,$$

can be compensated by the negative term, with no preferred scale.

Our (so far limited) work on distributions of supersymmetry breaking scales suggest that these are uniformly distributed. If so, these facts suggest that a high scale of supersymmetry breaking (probably in a hidden sector) is preferred.

[Home Page](#)[Title Page](#)[Page 29 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

Simple distributions for the gauge group and matter spectrum probably can be found as well. For example, one might hypothesize that the total rank of the gauge group is distributed as  $r^{-\alpha}$ , or the number of generations of matter is distributed as  $N^{-\beta}$ . Such hypotheses could be explicitly checked in brane constructions, or other classes of compactifications.

[Introduction](#)

[Vacua in . . .](#)

[Flux vacua](#)

[Distributions of . . .](#)

[Kähler moduli . . .](#)

[Conclusions](#)

[Home Page](#)

[Title Page](#)



Page 30 of 31

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

Simple distributions for the gauge group and matter spectrum probably can be found as well. For example, one might hypothesize that the total rank of the gauge group is distributed as  $r^{-\alpha}$ , or the number of generations of matter is distributed as  $N^{-\beta}$ . Such hypotheses could be explicitly checked in brane constructions, or other classes of compactifications.

Comparing these distributions between different dual classes of constructions (say type II and heterotic) will provide a highly non-trivial check of duality for  $N = 1$  and non-supersymmetric theories.

If two classes of construction produce the same statistics, that is evidence that both are representative of the statistics of the full ensemble of string/M theory vacua. We could then start to make interesting statements about the distribution of all the vacua.

[Home Page](#)[Title Page](#)[◀◀](#)[▶▶](#)[◀](#)[▶](#)[Page 30 of 31](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)

The most optimistic interpretation of such results will come out if it turns out that, on reaching the end of the problem and constructing fully consistent vacua, the total numbers of vacua are not too large, say of the order  $10^{100}$  that we wanted to solve the c.c. problem. At present, for this to come out, we must hope that many of the seemingly consistent tachyon-free vacua we just discussed in the end turn out to be unstable or inconsistent, or that some sort of cosmological selection applies.

In this case, such distribution results could imply that certain regions of moduli space in fact have **no vacua** which satisfy all the other constraints of the problem.

For example, if we found that approximately  $10^{40}$  vacua with a high scale of supersymmetry breaking) should match the SM, and  $10^{-40}$  with a low scale should match, we would have evidence for the claim that low scale supersymmetry breaking is not possible in string theory. Thus we might imagine making testable predictions which could falsify string theory.