LIGHT CONE GAUGE STRING FIELD THEORY AND AN INTEGRAL TRANSFORM

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In a physics problem described in the second half of this talk we (He, Spradlin, Volovich and JHS) encountered the integral equation

$$g(\zeta) = \int_0^\infty (\zeta^2+\mu^2)^{-3/2} f(\mu)\, d\mu \quad |\arg\zeta| < \pi/2,$$

where $g(\zeta)$ was a known function, and we needed to solve for $f(\mu)$. That paper proved that if g has appropriate analyticity and asymptotic properties, then

$$f(\mu) = \frac{\mu}{\pi i} \int_{-i\mu}^{i\mu} \frac{\zeta}{\sqrt{\mu^2+\zeta^2}}\, g(\zeta)\, d\zeta.$$

Letting $y = \mu^2$ and $z = \zeta^2$, as well as $F(y) = \frac{1}{2\mu} f(\mu)$ and $G(z) = g(\zeta)$, we obtain

$$G(z) = \int_0^\infty (y+z)^{-3/2} F(y)\, dy, \qquad |\arg z| < \pi$$

$$F(y) = \frac{1}{4\pi i} \int_{\mathcal{C}_y} \frac{G(z)}{\sqrt{y+z}} dz.$$

The function $G(z)$ has a branch cut running along the negative real axis and is holomorphic in the cut plane. The contour $\mathcal{C}_y$ starts and ends at $-y$ and encloses the origin in the counterclockwise sense. It is convenient to have a y-independent contour, so letting $z = wy$ and $\mathcal{C}_1 = \mathcal{C}$ we obtain

$$F(y) = \frac{1}{4\pi i}\sqrt{y}\int_{\mathcal{C}} \frac{G(yw)}{\sqrt{1+w}}\,dw,$$

An integration by parts gives

$$F(y) = -\frac{1}{2\pi i}\,y^{3/2}\int_{\mathcal{C}} \sqrt{1+w}\,G'(yw)\,dw.$$

I will now state and prove a generalization of the preceding result.

THEOREM

Let $F(y)$, which is only defined on the positive real axis, be a distribution (or "generalized function"), and let $\rho > 0$.

Suppose further that there exists a number α with $0 < \alpha < \rho$ such that $|\int_{y_1}^{y_2} y^{\alpha-\rho} F(y)\, dy|$ is bounded by a number independent of y_1 and y_2 for all $0 < y_1 < y_2$. Define

$$G(z) = \int_0^\infty (y+z)^{-\rho} F(y)\, dy, \qquad |\arg z| < \pi \tag{1}$$

Then $G(z)$ is holomorphic throughout the cut plane $|\arg z| < \pi$, and there exists a positive real number β such that $|z^\beta G(z)|$ is bounded at infinity. Moreover, the inverse transform is

$$F(y) = -\frac{1}{2\pi i} y^\rho \int_{\mathcal{C}} (1+w)^{\rho-1} G'(yw)\, dw. \tag{2}$$

The special case discussed earlier corresponds to $\rho = 3/2$.

The $\rho = 1$ case of eq. (1) is known as the Stieltjes transform, and the case of arbitrary ρ as the generalized Stieltjes transform (GST). The inverse GST in eq. (2) does not seem to have been found previously.

Much of the literature on the GST is concerned with determining the large $|z|$ asymptotic behavior of $G(z)$ for specified F. As it happens, in our application we were interested in deducing the asymptotic behavior of F associated with a given G.

In the special case $\rho = 1$ the transform in eq. (1) reduces to the Stieltjes transform

$$G(z) = \mathcal{S}_z[F] = \int_0^\infty \frac{F(y)}{y+z} dy.$$

This case provides an instructive test of our inverse transform. Setting $\rho = 1$ in eq. (2) gives

$$F(y) = -\frac{1}{2\pi i}\, y \int_{\mathcal{C}} G'(yw)\, dw$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\Big(G(-y - i\epsilon) - G(-y + i\epsilon)\Big).$$

Given the stated analytic and asymptotic properties of $G(z)$, it is a well-known fact that this is the inverse Stieltjes transform for $y > 0$.

The GST is related to the Laplace transform, for which we use the following notation

$$\mathcal{L}_x[F] = \int_0^\infty e^{-xy} F(y)\, dy.$$

Inserting the identity

$$(y+z)^{-\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty x^{\rho-1} e^{-x(y+z)} dx,$$

into eq. (1) gives

$$\begin{aligned} G(z) &= \frac{1}{\Gamma(\rho)} \int_0^\infty x^{\rho-1} e^{-xz} \mathcal{L}_x[F]\, dx \\ &= \frac{1}{\Gamma(\rho)} \mathcal{L}_z\left[x^{\rho-1} \mathcal{L}_x[F]\right]. \end{aligned}$$

In particular, setting $\rho = 1$, gives the well-known result that the Stieltjes transform is the operatorial square of the Laplace transform, i.e. $\mathcal{S} = \mathcal{L}^2$.

The previous equation implies that

$$\mathcal{L}_x[F] = \Gamma(\rho) x^{1-\rho} \mathcal{L}_x^{-1}[G],$$

where $\mathcal{L}^{-1}$ denotes an inverse Laplace transform. A second inverse Laplace transform gives

$$F(y) = \Gamma(\rho) \mathcal{L}_y^{-1} \Big[x^{1-\rho} \mathcal{L}_x^{-1}[G] \Big].$$

Our new result amounts to the statement that this can be simplified to take the form of eq. (2).

Let us now carry out some similar manipulations of eq. (2). Substituting

$$G'(yw) = -\int_0^\infty dt\, e^{-tyw} t\mathcal{L}_t^{-1}[G]$$

into eq. (2) and taking the Laplace transform of both sides recasts eq. (2) in the form

$$\mathcal{L}_x[F] = \Gamma(\rho+1)\frac{1}{2\pi i}\int_{\mathcal{C}} dw(1+w)^{\rho-1} \times \int_0^\infty dt(x+tw)^{-\rho-1} t\mathcal{L}_t^{-1}[G].$$

Comparing equations we deduce the identity

$$\frac{\rho}{2\pi i}\int_{\mathcal{C}} dw(1+w)^{\rho-1}(x+tw)^{-\rho-1}$$
$$= x^{-\rho}\delta(x-t). \qquad (3)$$

Another way of understanding the necessity of eq. (3) is to consider the special case $F(y) = \delta(y-t)$. In this case the GST is $G(z) = (t+z)^{-\rho}$. The inverse transform eq. (2) corresponds precisely to eq. (3) for this choice of $G(z)$.

Equation (3), which should hold for $x, t > 0$, is the heart of the matter.

PROOF OF EQ. (3)

We begin with the left side of eq. (3)

$$\chi_\rho(x,t) = \frac{\rho}{2\pi i}\int_{\mathcal{C}} dw(1+w)^{\rho-1}(x+tw)^{-\rho-1},$$

where $x, t > 0$. **For** $t > x$ **the contour can be pushed off to infinity, giving zero for the integral. For** $t < x$, **the contour encloses no singularity, so the result is again zero.**

We wish to prove that

$$\chi_\rho(x,t) = x^{-\rho}\delta(x-t).$$

Taking the Laplace transform of both

sides, we need to show that

$$\int_0^\infty dt\ \chi_\rho(x,t)e^{-ty} = x^{-\rho}e^{-xy}.$$

Substituting for $\chi_\rho(x,t)$ gives (after some simple manipulations)

$$\int_0^\infty dt\ \chi_\rho(x,t)e^{-ty}$$
$$= \frac{\rho}{2\pi i}y^\rho \int_{\mathcal{C}} dw(1+w)^{\rho-1}I(xyw), \quad (4)$$

where

$$I(u) = e^u \int_u^\infty e^{-v}v^{-\rho-1}dv.$$

If ρ is not an integer, this function can be rewritten in the form

$$I(u) = \Gamma(-\rho)\, e^{u} + u^{-\rho} I_2(u),$$

where $I_2(u)$ is regular at $u = 0$. The first term's contribution to eq. (4) is zero, since there is no singularity inside the contour. The function that matters is I_2, which satisfies the differential equation

$$u I_2'(u) - (u + \rho) I_2(u) + 1 = 0.$$

Substituting a power series expansion,

$$I_2(u) = \sum_{n=0}^{\infty} c_n u^n,$$

one obtains the recursion relation

$$(n+1-\rho)c_{n+1} = c_n.$$

Thus, since $c_0 = I_2(0) = 1/\rho$, **we conclude that**

$$c_n = \frac{\Gamma(1-\rho)}{\rho\,\Gamma(n+1-\rho)}.$$

To determine the contribution of each term to eq. (4), we need to evaluate

$$\frac{1}{2\pi i}\int_{\mathcal{C}} dw (1+w)^{\rho-1} w^{n-\rho}$$

$$= -\frac{\sin\pi(n-\rho)}{\pi} B(n-\rho+1,\rho) = \frac{(-1)^n}{\rho\, c_n\, n!}.$$

Combining these results, the c_n factors cancel, and we learn that

$$\int_0^\infty dt\, \chi_\rho(x,t)\, e^{-ty}$$
$$= x^{-\rho} \sum_{n=0}^\infty \frac{(-xy)^n}{n!} = x^{-\rho} e^{-xy},$$

which is the result that was desired.

Even though this derivation needs to be modified when ρ is a positive integer, the result is easily seen to be valid in that case as well. This completes the proof.

Let me now turn to the physics application of the formula for the inverse GST transform.

LIGHT CONE GAUGE STRING FIELD THEORY

1) Blau et al. constructed a maximally supersymmetric plane-wave background of type IIB string theory as a Penrose limit of the $AdS_5 \times S^5$ background.

2) Metsaev showed that IIB strings in this background are described in the GS light-cone gauge formalism by free massive bosons and fermions.

3) Berenstein, Maldacena, and Nastase identified the corresponding sector of SU(N) $\mathcal{N} = 4$ SYM and carried out some checks of the AdS/CFT duality in the appropriate limit.

4) Spradlin and Volovich generalized light-cone gauge superstring field theory to the plane-wave background.

These developments have aroused a great deal of interest, because they make it possible to compare perturbative gauge theory calculations with tractable string theory calculations.

To carry out these comparisons, one needs an expansion of the string field theory vertex in the limit that corresponds to the weakly coupled field theory. In terms of a mass parameter μ, to be defined, this is a large μ expansion, whereas flat space corresponds to $\mu = 0$.

The large μ analysis of the three-string vertex was worked out together with Y-H He, M. Spradlin, and A. Volovich in hep-th/0211198. The $AdS_5 \times S^5$ geometry is

$$ds^2(AdS_5) = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2)$$
$$ds^2(S^5) = R^2(\cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2)$$

Let $x^+ = t/\mu, \quad x^- = \mu R^2(\phi - t)$. Here μ is the mass scale and x^- has period $2\pi\mu R^2$. The conjugate momentum is $P_- = J/\mu R^2$ where J is an integer. $R \to \infty$ gives the plane-wave limit

$$ds_{pp}^2 = 2dx^+dx^- - \mu^2(x^I)^2(dx^+)^2 + dx^I dx^I$$

The duality correspondences are

$$\lambda = g_{YM}^2 N \quad \leftrightarrow \quad R^4/(\alpha')^2$$

$$g_{YM}^2 \quad \leftrightarrow \quad g_s$$

$$U(1)_R \quad \text{charge} \quad J \leftrightarrow \mu R^2 P_-$$

Setting $\alpha' P_- = 1$ **and letting** $R \to \infty$ **corresponds to** $J, N \to \infty$ **with finite parameters**

$$\lambda' = \frac{g_{YM}^2 N}{J^2} \leftrightarrow \mu^{-2}$$

$$g_2 = J^2/N \leftrightarrow \mu^2 g_s$$

λ' **is the effective gauge theory expansion parameter and** g_2 **is the genus-counting parameter. The goal is to analyze terms that are first order in** g_2 **and all orders in** λ'**.**

QUANTIZATION OF $H_{\ell c}$

Fourier analysis gives harmonic oscillators

$$[a_m^I, a_n^{J\dagger}] = \delta^{IJ}\delta_{mn} \qquad -\infty \leq m, n \leq \infty$$

$$I, J = 1, 2, ..., 8$$

with frequencies

$$\omega_n(\mu\alpha) = \sqrt{n^2 + (\mu\alpha)^2},$$

where $\alpha = \alpha' P_-$**. Then**

$$H_{\ell c} = \frac{1}{\alpha}\sum_{-\infty}^{\infty}\omega_n(a_n^{I\dagger}a_n^I + \text{ fermions})$$

Since the string action is in light-cone gauge ($x^+ = \tau$), this is both a spacetime and a world-sheet Hamiltonian.

Light-cone gauge string field theory was worked out for bosonic strings in 1973-74. In 1982-3 it was generalized to superstrings by Green, Brink, and JHS. The generalization to include mass parameter μ was carried out by Spradlin and Volovich .

Type IIB superstring field ($x^+ = \tau$):

$$\Phi(x^+, x^-, x^I(\sigma), \theta^a(\sigma))$$

Now transcribe first-quantized operators to second-quantized ones:

$$H_2 = i \int dx^- D^8 x(\sigma) D^8 \theta(\sigma) \partial_- \Phi H_{lc} \Phi, \text{ etc.}$$

and add interactions:

$$H = H_2 + H_3 + ..., \quad Q = Q_2 + Q_3 + ... \text{ etc.}$$

H_3 **describes the cubic vertex:**

$$H_3 = \int d\mu_3 G \tilde{\Phi}(1)\tilde{\Phi}(2)\tilde{\Phi}(3)$$

where

$$d\mu_3 = \prod_{r=1}^{3} d\alpha_r D^8 \lambda_r(\sigma) D^8 p_r(\sigma)$$

$$\cdot \delta(\sum \alpha_r) \Delta^8(\sum p_r(\sigma)) \Delta^8(\sum \lambda_r(\sigma))$$

and G **is an operator that acts at the interaction point** I **only. It is completely determined by the supersymmetry algebra (for flat space).**

The formulas are more precise when recast in terms of oscillators. This is needed to compute couplings of specific string states.

THREE STRING VERTEX

In the multi-Fock-space description it has the structure

$$|V_3\rangle = G E_a E_b |0\rangle$$

where

$$E_a = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} \bar{N}^{rs}_{mn} a_m^{(r)\dagger} a_n^{(s)\dagger}\right).$$

E_b **is a similar fermionic expression and** G **is polynomial in the various oscillators.** E_a **and** E_b **correspond to the** Δ **functionals and** G **to the interaction point operator.**

The Neumann matrices $\bar{N}^{rs}_{mn}$ **arise as products and inverses of various infinite matrices.**

The flat space ($\mu = 0$) **formulas are**

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha_1\alpha_2\alpha_3}{n\alpha_r + m\alpha_2}\bar{N}^r_m\bar{N}^s_n \qquad {\scriptstyle m,n=1,2,\ldots}$$

$$\bar{N}^r_m = \frac{1}{\alpha_r m!}\;\frac{\Gamma(m\gamma_r)}{\Gamma(1-m+m\gamma_r)}e^{m\tau_0/\alpha_r}$$

$$\tau_0 = \sum \alpha_r \log|\alpha_r|, \qquad \gamma_r = -\frac{\alpha_{r+1}}{\alpha_r}$$

The momenta may be scaled so that

$$\alpha_1 = y,\;\; \alpha_2 = 1-y,\;\; \alpha_3 = -1.$$

All quantities that we will compute are functions of two variables:

$$0 \le y \le 1 \qquad \text{and} \qquad 0 \le \mu < \infty$$

These functions are known for $\mu = 0$**.**

Fourier analysis of the overlaps with string 3 gives $(m, n = 1, 2, \ldots)$

$$A^{(1)}_{mn} = \frac{2}{\pi}(-1)^{m+n+1}\sqrt{mn}\frac{y \sin m\pi y}{n^2 - m^2 y^2}$$

$$A^{(2)}_{mn} = \frac{2}{\pi}(-1)^{m}\sqrt{mn}\frac{(1-y)\sin m\pi y}{n^2 - m^2 (1-y)^2}$$

$$A^{(3)}_{mn} = \delta_{mn}$$

We also define

$$C_{mn} = m\delta_{mn}$$

$$B_m = \frac{2}{\pi}(-1)^{m+1}\frac{\sin m\pi y}{y(1-y)m^{3/2}}$$

We also define (for $m, n = 1, 2, ...$) the following μ-dependent quantities

$$
\begin{aligned}
(C_r)_{mn} &= \sqrt{m^2 + (\mu\alpha_r)^2} \quad \delta_{mn} \\
(U_r)_{mn} &= [C^{-1}(C_r - \mu\alpha_r)]_{mn} \\
(\Gamma_+)_{mn} &= \sum_{r=1}^{3} (A^{(r)} U_r A^{(r)T})_{mn} \\
Y_m &= (\Gamma_+^{-1} B)_m \\
k &= B^T \Gamma_+^{-1} B.
\end{aligned}
$$

These are known for $\mu = 0$. For example,

$$
k(\mu = 0, y) = -2 \left(\frac{\log y}{1-y} + \frac{\log(1-y)}{y} \right).
$$

The goal is to evaluate

$$(\Gamma_+^{-1})_{mn}, \quad Y_m = (\Gamma_+^{-1} B)_m, \quad k = B^T \Gamma_+^{-1} B$$

as functions of y and μ and to expand them for large μ.

The first step is to express $(\Gamma_+^{-1})_{mn}$ in terms of Y_m and k:

$$\frac{m}{2\omega_m}\delta_{mn} + \frac{y(1-y)(\omega_m - \mu)(\omega_n - \mu) Y_m Y_n}{2[1 - \mu y(1-y)k](\omega_m + \omega_n)}$$

where

$$\omega_m = \sqrt{m^2 + \mu^2}.$$

Therefore if we knew Y_m and k we would know $(\Gamma_+^{-1})_{mn}$. Our strategy is to derive differential equations (in μ) and input the known values at $\mu = 0$.

Using the definitions given earlier and various identities satisfied by $A^{(r)}_{mn}$ and B_m, I derived the differential equation

$$\frac{\partial Y_m}{\partial \mu} = \left[\frac{1}{2}\frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) - \frac{\mu}{\omega_m^2}\right] Y_m$$

where

$$F(\mu, y) = \log[1 - \mu y(1-y)k(\mu, y)].$$

This has the solution

$$Y_m = \frac{m}{\omega_m} \exp\left[\frac{1}{2}\int_0^{\mu} \frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) d\mu\right] Y_m(0, y)$$

Thus, if we knew $F(\mu, y)$, we could deduce all the Neumann coefficients.

Since $Y_m \sim mB_m/(2\mu)$ for large μ, the preceding formula for $Y_m(\mu, y)$ implies that

$$\exp\left\{\frac{1}{2}\int_0^\infty \frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) d\mu\right\} = \frac{B_m}{2Y_m(0)}.$$

Taking the logarithm and integrating by parts gives

$$\int_0^\infty (m^2+\mu^2)^{-3/2} F(\mu, y) d\mu = G(m, y),$$

where

$$G(z, y) = \frac{2\tau_0}{z} + \frac{2}{z^2}\log\left(\frac{\Gamma(1+z)}{\Gamma(1+zy)\Gamma(1+z(1-y))}\right).$$

This formula must hold for $m = 1, 2, \ldots$ and $0 \leq y \leq 1$. The inverse integral transform determines $F(\mu, y)$.

For functions $G(z,y)$ that are holomorphic in the right half z plane with suitable asymptotics, we proved that the inverse integral transform is

$$F(\mu,y) = \frac{\mu}{\pi i}\int_{-i\mu}^{i\mu} \frac{\zeta}{\sqrt{\mu^2+\zeta^2}} G(\zeta,y)d\zeta.$$

Using this one can show that for our specific choice of $G(z,y)$

$$F(\mu,y) = -\ln[4\pi\mu y(1-y)] \\ +J(\mu y) + J(\mu(1-y)) - J(\mu),$$

where

$$J(x) = \frac{2}{\pi}\int_1^\infty \frac{\ln(1-e^{-2\pi x z})}{z\sqrt{z^2-1}}dz.$$

We can now derive asymptotic results that include all inverse powers of μ. For example, $F(\mu,y) \approx -\ln[4\pi\mu y(1-y)]$.

As an example, we considered the three-string vertex for the three states

$$\begin{aligned}
\langle 1| &= \langle 0| a^i_m a^j_{-m}, \\
\langle 2| &= \langle 0|, \\
\langle 3| &= \langle 0| a^i_n a^j_{-n},
\end{aligned} \tag{5}$$

where i and j are SO(4) indices.

We find to all finite orders in $1/\mu$ that

$$\langle 1|\langle 2|\langle 3|H\rangle \approx \frac{y(1-y)\sin^2(\pi n y)}{2\pi^2\sqrt{m^2+\mu^2 y^2}\sqrt{n^2+\mu^2}}.$$

This gives a prediction for the corresponding correlation function in the dual gauge theory. There is agreement at leading order (one loop), which is all that is known in the gauge theory.

HAPPY

BIRTHDAY

ALBERT