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Remarks on topological Landau-Ginzburg models

Based on work with Yi Li (Caltech) and Lev Rozansky (University of North Carolina).

$N=2$ $d=2$ susy field theory

$\Downarrow$ top. twist

2d TFT

$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$, where $Q^2 = 0$

$T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$ $\Rightarrow$ correlators of Q-closed ops. are g-independent

Q-exact ops. give 0-correlators

$\Rightarrow$ natural to work with Q-cohomology

3-pt correlators $\Longleftrightarrow$ (supercommutative) Frobenius algebra

$\langle\ ,\ \rangle$ (inner product)

$\langle a, bc \rangle = \langle ab, c \rangle$

$\langle\langle abc \rangle\rangle = \langle a, bc \rangle.$

$\uparrow$
correlator

More generally,

correlators of 2d TFT (in genus $g=0$) $\Longleftrightarrow$ Frobenius manifold

Ex 1. Take σ-model $\Sigma \to M$ where M is a Kähler manifold.

Twist gives "A-model".

Frobenius algebra: $\bigoplus_p H^p(X) \equiv H^*(X)$

but with a deformed product (quantum cohomology).

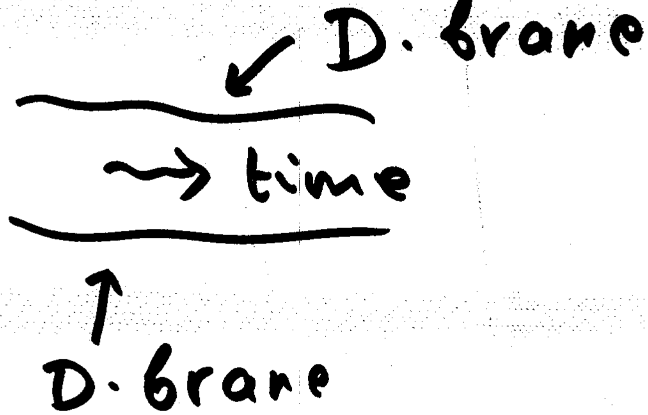
Ex 2. B-model (need $c_1(M)=0$)

Frobenius algebra: $\bigoplus_{p,q} H^p(\Lambda^q T^{1,0}X)$

Can we get NC Frobenius algebras?

yes!

Consider 2d TFT with bdries.

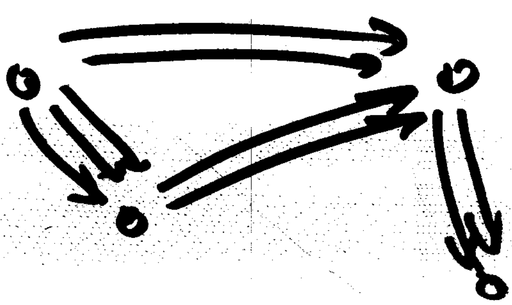


$\Rightarrow$ space of states $\mathcal{H}_0$

$Q: \mathcal{H}_0 \to \mathcal{H}_0$, $Q^2 = 0$.

$\Rightarrow$ Q-cohomology. V_0.

Get a vector space V_0 for every pair of D. branes.



Can compose arrows $\Rightarrow$ get a ($\mathbb{C}$-linear) category.

= End E (endomorphism algebra) is an NC Frobenius algebra.

More generally:

$$Hom(E,E') \otimes Hom(E',E) \to \mathbb{C}$$

(non-degenerate pairing).
= open-string metric

Ex 2. M, $c_1(M)$.
B-model with bdries.

$\Downarrow$

category of B-branes.

$\simeq$

$D^b(Coh\, M)$

Objects: complexes of vector bundles (or coherent sheaves).

Frobenius property = Serre duality theorem

Goal: identify categories of "topological D-branes" for other 2d TFTs.

Outline

- Topological LG models
- D-branes in topological LG models
- Reconstructing closed-string correlators from open ones: the Calabi-Yau case
- Reconstructing closed-string correlators from open ones: the LG case

Topological LG models

Let X be a (non-compact) Kähler manifold with $c_1(X) = 0$.

Let $W : X \to \mathbb{C}$ be a holomorphic function whose set of critical points ($dW = 0$) is compact.

LG-model is a 2d field theory with $N = (2, 2)$ SUSY and an action

$$S = \int d^2x\, d^4\theta\, K(\Phi, \Phi^\dagger) + \int d^2x\, d^2\theta\, W(\Phi) + h.c. \quad (1)$$

Φ is a chiral superfield $\Phi(x, \theta) = \phi(x) + \ldots$, where ϕ is a map from the worldsheet Σ to X.

The action has a $U(1)$ R-symmetry which can be used for topological twisting (of type B).

The A-twist is not possible, in general.

(c,c) ring of the LG model = BRST cohomology of the topological LG model.

Most common case: $X = V \simeq \mathbb{C}^n$. The BRST cohomology is

$$V_c = \mathbb{C}[V]/I_W, \quad I_W = (\partial_1 W, \ldots, \partial_n W).$$

Another popular case: $X = V/G$, where G is a finite group. BRST cohomology was computed by Intriligator and Vafa (1990).

V_c has a natural scalar product (topological metric), which for $X = V$ has been computed by Vafa (1991):

$$\langle f, g \rangle = \mathrm{tr}(fg), \quad f, g \in \mathbb{C}[V]/I_W,$$

where

$$\mathrm{tr}(f) = \frac{1}{(2\pi i)^n} \oint \frac{f dz^1 \wedge \ldots \wedge dz^n}{\partial_1 W \ldots \partial_n W} \tag{2}$$

Contour: $|\partial_i W| = \epsilon_i$.

In other words, V_c is a commutative Frobenius algebra.

This structure encodes tree-level 3-point correlators in top. LG-model coupled to top. gravity. To get higher-point correlators, one has to enhance this to a Frobenius manifold.

B-branes in LG models

The set of D-branes in any TFT carries the structure of a Differential Graded (DG) category, or more generally of an A_∞ category.

From now on we mostly let $X = V$.

Maxim Kontsevich has formulated a conjecture about the mathematical description of the category of B-branes in the LG model.

Objects: $\mathbb{Z}_2$ graded free modules of finite rank over $A = \mathbb{C}[V]$ equipped with an odd endomorphism $D : E \to E$ satisfying $D^2 = W \cdot id_E$.

Morphisms: the set of morphisms from (E, D) to (E', D') is a $\mathbb{Z}_2$-graded vector space $Hom_A(E, E')$ with a differential

$$\mathcal{D} : \phi \mapsto D' \circ \phi - (-1)^{\phi} \phi \circ D.$$

$\mathcal{D}$ is an odd endomorphism satisfying $\mathcal{D}^2 = 0$. Its cohomology is identified with the BRST cohomology V_o of the open-string Hilbert space.

This construction gives a DG-category, i.e. spaces of morphisms are $\mathbb{Z}_2$-graded complexes of vector spaces. Taking the cohomology of these vector spaces, we get a triangulated category.

If $X = V/G$, we can define the category of B-branes in the same way, except that all the objects involved must be equivariant with respect to the G-action.

Kontsevich's conjecture was explained in physical terms by

A.K. and Yi Li (hep-th/0210296, hep-th/0305136)

Brunner et al. (hep-th/0305133).

See also Lazaroiu (hep-th/0312286).

The idea is to use both branes and anti-branes to write down a BRST-invariant boundary condition. The boundary action depends on the boundary tachyon T, which is a sum of holomorphic and anti-holomorphic pieces:

$$T = F + G^\dagger, \quad F \in Hom(E, E'), \quad G \in Hom(E', E).$$

Define

$$D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$$

BRST-invariance requires $FG = GF = W$, which is the same as $D^2 = W$.

Consider for simplicity open strings which beging and end on the same In the zero-mode approximation, the space of states is the space of smooth sections of $\Omega^{0,\star}(\mathrm{End}E)$.

The BRST operator is $Q = \bar{\partial} + \mathcal{D}$.
Q-cohomology is the same as $\mathcal{D}$-cohomology on *holomorphic* sections.

A simple example

$$W = z^n$$

Simple B-branes: $E_k = \mathbb{C}[V] \oplus \mathbb{C}[V]$,

$$D_k = \begin{pmatrix} 0 & z^{n-k} \\ z^k & 0 \end{pmatrix}.$$

Cohomology of $\mathcal{D}$ is isomorphic to an algebra with two generators: x (bosonic) and θ (fermionic) and relations

$$x^p = 0, \theta^2 = -x^{n-2p}, \quad p = min(k, n-k).$$

For $k = 0$ and $k = n$ the brane is trivial, so $0 < k < n$.

One can show that any B-brane is isomorphic to a finite sum of these basic branes.

The simplicity of this example is misleading: the category of B-branes has a finite number of irreducible objects if and only if W defines a minimal model.

Topological open-string metric

Axioms of 2d TFT imply that the open-string BRST cohomology is a Frobenius algebra (in general, noncommutative).

The open-string trace was derived by A.K. and Yi Li (hep-th/0305136) (see also Herbst and Lazaroiu (hep-th/0404184)):

$$\mathrm{Tr}\,\phi = \frac{1}{n!(2\pi i)^n} \oint \frac{\mathrm{STr}\left((\partial D)^{\wedge n}\,\phi\right)}{\partial_1 W \ldots \partial_n W}$$

More generally, there is a bulk-boundary map $V_o \to V_c$ which is given by

$$\phi \mapsto \frac{1}{n!}\mathrm{STr}\left((\partial D)^{\wedge n}\,\phi\right)$$

The r.h.s. is a polynomial which is regarded as an element of $\mathbb{C}[V]/I_W$.

Closed strings from open strings: the CY case

It has been conjectured by M. Kontsevich (1994) that closed-string space of states (for topological string theories) is the Hochschild cohomology of the category of D-branes.

- Hochschild cohomology of an algebra A (with coefficients in A) computes the space of infinitesimal deformations of A regarded as A_∞ algebras (M. Penkava and A. Schwarz, 1994).
- Notation: $H^*(A, A)$, or simply $H^*(A)$.
- $H^*(A)$ is the cohomology of a certain differential δ acting on the vector space

$$\oplus_{n=0}^{\infty} A^{\otimes n}.$$

- $H^*(A)$ is a supercommutative algebra for any A.

- Hochschild cohomology can be defined for DG-algebras, and more generally for A_∞ algebras.

- Hochschild cohomology can also be defined for sheaves of algebras.

- If the category of D-branes is the category of modules over an algebra, or a sheaf of algebras, one may define its Hochschild cohomology as the Hochschild cohomology of this algebra.

- More generally, one can either think of a category $\mathcal{A}$ as an algebra with many objects, or define the Hochschild cohomology as the endomorphism algebra of the identity functor from $\mathcal{A}$ to itself (in the category of functors between A_∞ categories).

The latter point of view leads to a geometric definition of $H^*(\mathcal{A})$ for the category of B-branes on a CY X. The diagonal $\Delta \subset X \times X$ represents the identity functor. On the other hand, it is a B-brane on $X \times X$, i.e. an object of $D^b(Coh(X \times X))$. Hence one can define $H^*(\mathcal{A})$ as the endomorphism algebra of Δ in $D^b(Coh(X \times X))$. Conjecturally, this is the same as the endomorphism algebra in the functor category.

This definition gives the right result for the space of closed-string states:

$$H^*(\mathcal{A}) = \oplus_{p,q} H^p(\Lambda^q TX).$$

This is precisely the space of states of the B-model on X.

Closed strings from open strings: LG case

We need to understand the analogue of $\Delta \subset X \times X$ for LG models.

• The LG analogue of $X \times X$ is the LG model with target $X \times X = \mathbb{C}[V \oplus V]$ and superpotential

$$W(x) - W(y).$$

The analogue of Δ is a certain B-brane Δ_W on $X \times X$. For $X \simeq \mathbb{C}$ it is a trivial rank-two bundle $E \simeq \mathbb{C}[x,y] \oplus \mathbb{C}[x,y]$ with

$$D = \begin{pmatrix} 0 & \frac{W(x)-W(y)}{x-y} \\ x-y & 0 \end{pmatrix}$$

• If we set $W = 0$, then the pair (E, D) becomes a two-term complex, whose cohomology is the structure sheaf of the diagonal Δ. (More precisely, (E, D) becomes the Koszul resolution of Δ).

• The object Δ_W represents the identity functor in the category of D-branes (Khovanov and Rozansky, hep-th/0401268).

Computing the endomorphism algebra of Δ_W, we get

$$H^*(\mathcal{A}) = \mathbb{C}[x, y]/I,$$

where

$$I = \left(x - y, \frac{W(x) - W(y)}{x - y} \right).$$

Obviously, this is the same as $\mathbb{C}[x]/I_W$.

One can extend this to the multi-variable case.

One can also show that the open-string trace on the endomorphism algebra of Δ_W coincides with the closed-string trace on $\mathbb{C}[x]/I_W$. Thus we can recover the Frobenius algebra structure on V_c by regarding it as the Hochschild cohomology of the category of B-branes.

Hopefully, one can recover the whole Frobenius *manifold*.

All this seems to work for orbifolds V/G. For orbifolds, V_c contains contributions both from the untwisted sector (the invariant part of $\mathbb{C}[V]/I_W$) and the twisted sector.

E.g. consider $X \simeq \mathbb{C}^3$, $W = x_1^3 + x_2^3 + x_3^3$ and $G = \mathbb{Z}_3$ acting by

$$x_i \mapsto \omega x_i, \quad i = 1, 2, 3, \quad \omega^3 = 1.$$

This is a Gepner model for an elliptic curve. Hence the space of closed-string states is 4-dimensional, with 2-dimensional even subspace.

The diagonal brane $\Delta_{W,G}$ is the equivariantized version of Δ_W (with respect to the $G \times G$ action). We checked that its endomorphism algebra is isomorphic to the B-model of the elliptic curve as a Frobenius algebra.

We repeated this exercise for other Gepner models, and found the expected results.

Concluding remarks

- It seems that for topological string theories the closed-string TFT can be recovered from the Hochschild cohomology of the category of D-branes, at least on the Frobenius algebra level.

- The bulk-boundary map can also be recovered in abstract terms, by noting that the endomorphism algebra of any D-brane is a bi-module over the Hochschild cohomology.

- It looks plausible that all open-closed string correlators can be constructed in a similar way, by using only the A_∞ structure on the category of topological D-branes.

- The only essential property required from an A_∞ category seems to be the existence of a non-degenerate pairing between spaces of morphism in opposite directions, for any two objects. In mathematical terms, one needs to have a Serre functor which is trivial (up to a shift of grading).