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Discrete Connections on the Triangulated Manifolds and difference linear equations

Abstract: *Following the authors works [1, 2, 3], we develop a theory of the discrete analogs of the differential-geometrical GL_n-connections over the triangulated n-manifolds. We study a nonstandard discretization based on the interpretation of DG Connection as the linear first order ("triangle") difference equation in the simplicial complexes acting on the scalar functions of vertices. This theory appeared as a by-product of the new type of discretization for the special Completely Integrable Systems, such as the famous 2D Toda Lattice and corresponding 2D stationary Schrodinger operators. A nonstandard discretization of the 2D Complex Analysis based on these ideas was developed in the recent work [4]. A complete classification theory is constructed here for the Discrete DG Connections based on the mixture of the abelian and nonabelian features.*

I.General Definitions: The Discrete DG Connections.

Let M be a n-dimensional simplicial complex.

By the **Discrete Differentially-Geometrical (DG) Connection** we call any set of coefficients $0 \neq b_{T:P} \in k^*, k = R, C$, assigning a nonzero number to every pair consisting of the n-simplex T and its vertex $P \in T$.

Every DG-connection defines a first order difference **Triangle Operator** Q mapping the space of k-valued functions of vertices ψ_P into the space of functions of n-simplices:

$$(Q\psi)_T = \sum_{P \in T} b_{T:P}\psi_P$$

Such operators played an important role in the works [1, 2, 3]. For the needs of the theory of discrete DG Connections only the linear **Triangle**

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Equation is important

$$Q\psi = 0$$

well defined up to the **Abelian Gauge Transformations**

$$Q \to f_T Q g_P^{-1}, \psi_P \to g_P \psi_P$$

where $f \neq 0, g \neq 0$. Beginning from now we denote vertices by the letters $i, j, l, \ldots$.

Therefore for every n-simplex T with vertices $i, j \in (i_0, \ldots, i_n)$ only the ratios are essential $\mu_{ij}^T = b_{T:i}/b_{T:j}$ where $b_{T:i}$ are the coefficients of DG connection associated to the vertices of the simplex T. We assume beginning from now that **the DG Connection is given by the set of nonzero numbers μ_{ij}^T for all n-simplices and pairs of their vertices.** Obviously we have

$$\mu_{ii}^T = 1, \mu_{ij}^T \mu_{ji}^T = 1, \mu_{ij}^T \mu_{jk}^T \mu_{ki}^T = -1$$

.

Let T, T' be a pair of n-simplices such that $i, j \in T \bigcap T'$. We define a **Gauge-Invariant Coefficients**

$$\mu_{ij}^T \mu_{ji}^{T'} = \rho_{ij}^{TT'}$$

Lemma 1 *The whole set of the gauge invariant coefficients $\rho_{ij}^{TT'}$ can be recovered from the* **Minimal Subset** *such that T and T' are the closest neighbors, i.e. the intersections $T \bigcap T'$ are the $(n-1)$-dimensional faces. There are "trivial" sets A and B of relations on these quantities:*

A.For every triangle $[ijl] \subset T \bigcap T'$ we have

$$\rho_{ij}^{TT'} \rho_{jl}^{TT'} \rho_{li}^{TT'} = 1$$

B.For every closed path $T_0 T_1 \ldots T_N T_0$ in the Poincare dual cell subdivision of the triangulated manifold M where n-simplices T define the vertices, all pairs $T_p \bigcap T_{p+1}$ define the edges (they are dual to the $n-1$-faces), and the edge $< ij >$ belongs to all T_p, we have

$$\prod_{p=1}^{N} \rho_{ij}^{T_p T_{p+1}} = 1$$

Proof. For the simplicial manifold M every pair of n-simplices T, T' such that $i,j \in T \bigcap T'$ can be joined by "path" $T_0 = T, T_1, \ldots, T_m = T'$ where $i,j \in T_k \bigcap T_{k+1}$ for all $k = 0, \ldots, m-1$, and $T_k \bigcap T_{k+1}$ are $(n-1)$-simplices for all values of k. We have by definition

$$\rho_{ij}^{TT'} = \prod_{k=0}^{k=m-1} \rho_{ij}^{T_k T_{k+1}}$$

Our trivial set A of the relations for these quantities follows from the same relations for the quantities $\mu_{ij}^{T}, \mu_{ij}^{T'}$ as above. In order to prove the set of relations B we point out that any such closed path in the dual cell decomposition can be obtained as a product of elementary paths $T_0 \ldots T_m$ corresponding to the simplicial stars of every $n-2$-simplices $< ij > \subset \sigma \subset St(ij)$. We have here

$$St(\sigma) = T_0 \bigcup \ldots \bigcup T_m$$

and the relation

$$\rho_{ij}^{T_0 T_1} \ldots \rho_{ij}^{T_m T_0} = \mu_{ij}^{T_0} \mu_{ji}^{T_1} \mu_{ij}^{T_1} \ldots \mu_{ji}^{T_0} = 0$$

Lemma is proved.

Problem: Is it possible to recover the whole DG connection from the Minimal Subset of the abelian gauge-invariant coefficients $\rho_{ij}^{TT'}$? Which invariants of DG connection should be added if it is impossible?

We are going to solve this problem for the 2D and 3D manifolds $n = 2, 3$ where the whole set of the additional invariants is easy to find out: let us choose any set of the closed combinatorial "framed" paths $\gamma_1, \ldots, \gamma_{b_1}, a_1, \ldots, a_{tor_1}$ representing the basis of the homology group $H_1(M, Z)$. We define the following abelian gauge-invariant **Topological quantities**:

$$\mu(\gamma) = \prod_{\gamma} \mu_{l,l+1}^{T_l}$$

for every closed "framed path" γ_k consisting of edges through the vertices $0, 1, \ldots, l, l+1, \ldots, m_k = 0$, equipped by such triangles (n-simplices) T_l that $[l, l+1] \subset T_l$.

We are going to prove below the following

Theorem 1 *For any $n \geq 2$ the set of invariants*

$$\{\rho_{ij}^{TT'}, \mu(\gamma_k), \mu(a_s)\}$$

is complete where $i \neq j, \rho_{ij}^{TT'}, [ij] \subset T \bigcap T'$.

For the compact oriented 2-manifolds the only nontrivial relation on these quantities is

$$\prod_{[ij]\in M} \rho_{ij}^{TT'} = 1, \partial T = [ij] + ..., \partial T' = [ji] + ..., T \bigcap T' = [ij]$$

Here $[ij]$ *means all edges in the manifold* M*, the 2-simplices* T, T' *are oriented as prescribed by the global orientation.*

For the compact oriented n-manifolds the complete set of relations on these quantities can be described in the following way: there are trivial relations A *for every 2-simplex* $[ijl] = \Delta \subset T \bigcap T'$ *where*

$$\rho_{ij}^{TT'} \rho_{jl}^{TT'} \rho_{li}^{TT'} = 1$$

and the relations B *for every closed path in the Poincare dual cell decomposition corresponding to the boundaries of the dual 2-cells. For the description of the nontrivial relations we choose the set of integral 2-chains*

$$z_1, ..., z_{b_2}, u_1, ..., u_{tor_1}\}$$

where $z_1, ..., z_{b_2}$ *is the basis of the group* $H_2(M, Z)$ *and* u_s *represent the basis of cycles mod* m_s*, i.e.* $\partial u_s = m_s a_s$*. Let all chains* z, u *be presented as sums* $\sum \Delta_k$ *with "framing"* T_k *for every oriented 2-simplex* Δ_k*,* $\partial T_k = \Delta_k + \ldots$*. The nontrivial relations have the following form: The product*

$$\prod_{[ij]=\Delta_k \bigcap \Delta_{k'}} \rho_{ij}^{T_k T_{k'}}$$

is equal to one for the cycles z*; it is equal to* $\mu(\partial u_s)^* = \mu(a_s)^{m_s}$ *for the torsion part* u_s*.*

For $n = 3$ *exactly one global relation between these relations follows from the following identity:*

$$\prod_{[ij]\in M} \rho_{ij}^{TT'} = 1$$

in the compact oriented 3-manifold M *where orientation in the 3-simplices* T, T' *is induced by the one in* M, $\partial T = [ijl], \partial T' = [jil]$.

For $n \geq 2$ *any set of quantities*

$$\rho_{ij}^{TT'}, \mu(\gamma_k), \mu(a_s), k = 1, ..., b_1, s = 1, ..., torsion_1$$

satisfying to this set of trivial and nontrivial relations can be realized by the discrete DG Connection uniquely up to the abelian gauge transformation.

II. The Nonabelian Curvature.

By definition, **the Nonabelian Curvature** for the discrete GL_n Connection of the type described above is the obstruction to the existence of full n-dimensional space of the local solutions to the triangle equation $Q\psi = 0$ in the whole simplicial stars of the vertices. However, on the manifold M it is enough to consider only the obstructions to the existence of local solutions in **the simplicial stars $St(\sigma)$ for all $(n-2)$-simplices** $\sigma = [0, 1, \ldots, n-2]$. The whole set of vertices in this star contains also the complementary set $p, p = 1, 2, \ldots, m$ where the number m depends on σ. The n-simplices in this star are exactly the following

$$T_p = [\sigma, p-1, p]$$

where p is counted here modulo m. Every n-simplex T_p contains in its boundary a pair of faces inside of this star:

$$[\sigma, p-1] \bigcup [\sigma, p] \subset \partial T_p$$

We start with initial data taking $\psi_0, ..., \psi_{n-2}, \psi_p$ as an arbitrary n-vector η. From the equation $Q\psi = 0$ in the simplex T_{p+1} we obtain the value

$$\psi_{p+1} = \sum_{q=0}^{q=n-2} \mu^{T_{p+1}}_{q,p+1} \psi_q + \mu^{T_{p+1}}_{p,p+1} \psi_p, q \in \sigma, p = 1, ..., m$$

$$\psi_q = \psi_q, q = 0, ..., n-2$$

or $\eta \rightarrow A_p(\eta) = \eta'$ where A_p is a lower triangle matrix with $(1, ..., 1, \mu^{T_{p+1}}_{p,p+1})$ along the diagonal. It has only the last nontrivial row except diagonal which is equal to $(\mu_{0,p+1}, ...\mu_{n-2,p+1}, \mu_{p,p+1})$. For simplicity we omitted in these formulas the simplices σ, T_{p+1}. Their presence is assumed. The full cyclic product of these matrices gives us by definition **a Nonabelian Curvature Operator** around the $(n-2)$-simplex σ:

$$K_{\sigma,p} = A_{p+m-1} A_{p+m-2} \cdots A_{p+1} A_p$$

of the same algebraic form as all matrices A_s. Here s is counted modulo m. Its diagonal part is equal to

$$1, ...1, \mu_\sigma = \prod_{s=p}^{s=p+m-1} -\mu_{s,s+1}^{T_{s+1}} = \det K_{\sigma,p}$$

Its last row is equal to

$$\alpha_{\sigma,0,p}, ..., \alpha_{\sigma,n-2,p}, \mu_\sigma$$

We call coefficients $\alpha_{\sigma,q,p}$ and μ_σ **the Local Curvature Coefficients** where $q \in \sigma, p \in (1, 2, ..., m)$. We say that our Discrete DG Connection is **Locally Unimodular, i.e. belongs to the group** SL_n if $\det K_(\sigma) = \mu(\sigma) = 1$ for all σ. The Connection is **Locally Flat** if $K_\sigma = 1$ for all σ. These definitions imply immediately the following

Lemma 2 *The local curvature operators are equal to the unit matrix if and only if our linear equation $Q\psi = 0$ has exactly n-dimensional space of solutions on the universal covering space. Its coefficients can be computed by formula*

$$\alpha_{\sigma,q,p} = \mu_{q,p}^{T_p} + \mu_{p-1,p}^{T_p}\mu_{q,p-1}^{T_{p-1}} + ... + \mu_{p-1,p}^{T_p}\mu_{p-2,p-1}^{T_{p-1}} \cdots \mu_{p-m+1,p-m+2}^{T_{p-m+2}}\mu_{q,p-m+1}^{T_{p-m+1}}$$

In particular these coefficients transform as multiplicative 1-chains under the abelian gauge transformations

$$\psi_j = h_j\phi_j, h_j \neq 0, \psi_i \to \phi_i, \mu_{ij}^T \to (h_i/h_j)\mu_{ij}^T$$

$$\mu_\sigma \to \mu_\sigma, \alpha_{\sigma,q,p} \to (h_q/h_p)\alpha_{\sigma,q,p}$$

We can see now how to organize the simplest gauge invariant expressions.

Lemma 3 *The quantities*

$$\alpha_{\sigma,q,p}\mu_{p,q}^{T_p} = \alpha_{\sigma,q,p}^*$$

are gauge invariant. They are connected with each other by the formula

$$\alpha_{\sigma,q,p+1}^* = -\alpha_{\sigma,q,p}^*/(\mu_{pq}^{T_p}\mu_{qp}^{T_{p+1}}) + (1 - \mu_\sigma)$$

All quantities $\alpha^*_{\sigma,q,p}$ *and* μ_σ *can be expressed through the gauge invariant coefficients* $\rho^{TT'}_{ij} = \mu^T_{ij}\mu^{T'}_{ji}, T, T' \in St(\sigma)$, *by the formula*

$$\alpha^*_{\sigma,q,p} = \sum_{k=0}^{k=m-1} (-1)^k \prod_{j=0}^{j=k} \rho^{T_{p-j+1},T_{p-j}}_{p-j,q}$$

$$\prod_{p=1}^{p=m} \rho^{T_p,T_{p+1}}_{qp} = (-1)^m \mu_\sigma, q = 0, 1, ..., n-2$$

so we have $n-1$ *different expressions for the same quantity* μ_σ.

Corollary 1 *For the locally unimodular (i.e. locally* SL_n*) connections the condition* $\alpha_{\sigma,q,p} = 0$ *does not depend on the initial point* $p \in ST(\sigma)$, *so the property to be locally flat in the star* $ST(\sigma)$ *depends on the* $n-2$*-simplex* σ *only.*

Corollary 2 *For the generic connections such that* $\alpha_{\sigma,p} \neq 0$ *for all* σ, p, *all data* $\rho^{TT'}_{ij}$ *can be reconstructed from the gauge invariant coefficients of the local nonabelian curvature* $\alpha^*_{\sigma,q,p}, \mu_\sigma$.

Proof of the lemma. Starting with the equation expressing $\alpha_{\sigma,q,p}$ through the collection of μ-s in the previous lemma, we multiply both its sides by the quantity $\mu^{T_p}_{p,q}$. After the elementary manipulations, we are coming to the expressions for α^* which easily implies all statements of this lemma. Let us avoid here these elementary calculations.

For the proof of the first corollary, we point out that the condition $\mu_\sigma = 1$ implies that

$$\alpha^*_{\sigma,q,p+1} = \alpha^*_{\sigma,q,p}/\rho^{T_pT_{p+1}}_{p,p+1}$$

where always $\rho \neq 0$, so our corollary obviously follows.

In order to prove second corollary, let us point out that under the assumption of this corollary, we represent ρ as a ratio of the nonzero numbers

$$\rho^{T_pT_{p+!}}_{p,p+1} = \alpha^*_{\sigma,q,p}/\{\alpha^*_{\sigma,q,p+1} - 1 + \mu_\sigma\}$$

This formula proves our corollary.

III.The Abelian (Framed) and Nonabelian Holonomy Groups.

The **Abelian Framed Holonomy Representation** we define for the **Framed Combinatorial Paths** starting and ending in the same point. By definition, a **Framed Combinatorial Path** is a sequence of edges equipped by the n-simplices containing these edges

$$\gamma =< i_0, i_1, ..., i_k, i_{k+1}, ..., i_N = i_0 | T_0, T_1, ..., T_k, ..., T_{N-1} >$$

where $[i_k, i_{k+1}] \subset T_k$. The Abelian Framed Holonomy $\mu(\gamma)$ for the framed path γ is equal to the product

$$\mu(\gamma) = \prod_k (-\mu^{T_k}_{i_k, i_{k+1}})$$

There is a natural multiplication of the framed combinatorial paths $\gamma_1\gamma_2$ and a whole associative semigroup of them $\Omega = \Omega_{fr}(M, i_0)$. We have an **Abelian Framed Holonomy Representation**

$$\Omega_{fr}(M, i_0) \to k^*$$

For every framed path there is a natural **Inverse Framed Path** γ^{-1} consisting of the same edges and n-simplices but the order of passing them is reversed. The inverse framed path leads to the inverse framed holonomy. We factorize this semigroup by the relations

$$< i, i | T >=< i, j | T >< j, i | T >= 1$$

$$< ij | T >< jk | T >< ki | T >= 1$$

for the vertices i, j, k belonging to the same n-simplex T. Our relations mean exactly that such pieces can be removed from any path if you meet these vertices one after another as the closest neighbors. We call factor by these relations of the semigroup $\Omega_{fr}(M, i_0)$ by the **Framed Fundamental Group**

$$\pi_1^{fr}(M, i_0) = \Omega_{fr}(M, i_0) // (Relations)$$

For the Abelian Framed Holonomy we need only the factor-group by the commutation relations

$$\pi_1^{fr}(M, i_0) // (aba^{-1}b^{-1}) = H_1^{fr}(M)$$

We call this factor a **Framed Homology Group** written in the multiplicative form.

Lemma 4 *The framed homology group is generated by the framed paths* $< i, j, i|T, T' >$ *and by the arbitrary closed framed paths* γ_s *whose image in the ordinary homology group (the unframed part) generates it.*

Remark 1 *For the choosing generators of the framed fundamental group we have to fix for every vertex* j *a framed path* δ_i *joining the vertices* 0 *and* i*. As usually, we consider the set of closed paths* $(\delta_i) < i, j, i|TT' > (\delta_i^{-1})$ *as the additional generators in the framed fundamental group. In this work we are dealing with the abelian case only.*

The Abelian Framed Holonomy leads to the representation

$$H_1^{fr}(M) \to k^*$$

The ordinary homology we obtain as a factor-group

$$H_1^{fr}//(< ij|T >< ji|T' >) = H_1(M)$$

with factorized holonomy representation

$$\mu : H_1(M) \to k^*//\{\rho_{ij}^{TT'}\}$$

for all i, j, T, T'.

In order to define a **Full Nonabelian Holonomy Representation** we introduce an important notion of a **Thick Path** as a sequence of the oriented n-simplices $\kappa =< T_1, ..., T_N >$ such that the next one is attached to the previous one along the common $n-1$-dimensional face Δ where they induce the opposite orientations. For the **Irreducible Thick Path** the intersections $T_k \bigcap T_{k+1} = \Delta_k$ should be exactly equal to the $n-1$-dimensional faces for all $k = 1, ..., N-1$, i.e. $T_k \neq T_{k+1}$. By definition **the Closed Thick Paths with period** N are defined by the condition that the last $n-1$-dimensional out-simplex Δ_N coincides with the initial simplex Δ_0. There is also a notion of the **Periodic Thick Paths** where these sequences are infinite and periodic. We can obviously multiply closed thick paths with the same initial and final $n-1$-simplex Δ_0. The notions of the inverse thick path and of the trivial (empty) thick path are natural. Therefore we are coming to the **Associative Semigroup of the Closed Thick Paths** $\Omega^{thick}(M, \Delta_0^{n-1})$.

Let us construct a geometric model of the **Abstract Thick Paths**. We start with the standard linear $n-1$-simplex $\Delta_0 = [0, 1, ..., n-1] \subset R^{n-1}$

multiplied by the real line R going along the n-th axis x_n. Our abstract thick path κ_A will be defined by the word $A = a_{i_q}^{r_q}...a_{i_0}^{r_0}$ of any length $N = \sum_{s=0}^{s=q} r_s$ in the free associative semigroup with n generators $a_0, ..., a_{n-1}$. As a first step, we take a vertex i_0 of the initial $n-1$-simplex Δ_0 for $x_n = 0$ and shift it along the nth axis on the positive distance. We get new vertex i'_0. Now we construct a linear n-simplex T_1 with vertices $[0, ..., n-1, i'_0]$. It contains in the boundary an original **in-simplex** $\Delta_0 = [0, ..., n-1]$ and a new linear $n-1$-simplex $\Delta_1 = [0, ..., i'_0, ...n-1]$ (the first **out-simplex**) where exactly one vertex i_0 is replaced by the shifted one i'_0. Now taking the out-simplex Δ_1 as a new in-simplex instead of Δ_0, we perform this operation once more: we take one of the vertices of the out-simplex $i'_1 \in \Delta_1$ and shift it along the corresponding axis up on the level higher than i'_0. It may be the same vertex (it should appear exactly r_0 times here as in the word A), or another one if we already passed all r_0 steps. After that we construct a linear $n-1$-simplex Δ_2 as a out-simplex for the n-simplex T_2 and so on. Finally we are coming to the realization of the whole word A as a "prism" over Δ_0 consisting of the n-simplices such that all their vertices are located on the x_n-lines R_k over the original vertices $k \in \Delta^{n-1}$ with monotonically increasing heights x_n except of the vertices of the initial Δ_0. This is an abstract model of thick path with combinatorics defined by the word A consisting of the linear n-simplices with vertices located in the union of the n "angle lines" only. For every number $k = 0, 1, ..., n-1$ there is a subsequence of n-simplices in the thick path $T_l^{(k)} \subset \kappa_A$ with shifts up along the coordinate x_n over the vertex with number k in the vertices $j_l = l \in R_k, l = 0, 1, 2, ..., N_k$ such that $\sum_{k=0}^{k=n-1} N_k = N$, and $N_k = \sum_s r_s$ where $i_s = k$. We call these sequences $< j_0, j_1, ..., j_{N_k} | T_0^{(k)}, ..., T_{N_k}^{(k)} >$ **the abstract angle framed paths**.

Any thick path can be realized in the manifold M with the initial oriented $n-1$-simplex Δ_0 is given. An initial oriented simplex Δ_0 uniquely determine the irreducible thich path with given combinatorics. All n-simplices T_k should be attached to the previous oriented $n-1$ out-simplex Δ_{k-1} inducing in it the right orientation, and $T_k \neq T_{k-1}$ in the irreducible case. For the realization of the reducible thick path we should indicate the "turning points" in the sequence of simplices. An irreducible thick path will be determined by the combinatorics of the word A only. Topology of the triangulated manifold M determines which paths with the initial face $\Delta = \Delta_0$ are in fact closed. **The set of closed thick paths started in the face Δ_0 we denote by**

$\Omega^{thick}(M, \Delta_0)$. For the closed thick path the corresponding angle framed paths will not necessarily be closed in the manifold M:

Lemma 5 *There is a natural homomorphism into the permutation group S_n*

$$P : \Omega^{thick}(M, \Delta_0) \to S_n$$

induced by the permutation of the vertices of the $n-1$-simplex after the identification of the initial in-simplex Δ_0 and the last out-simplex in the closed thick path.

The kernel of this representation will be denoted $\ker P = \Omega_0^{thick}(M, \Delta_0)$. Let a closed thick path $\kappa \in \Omega^{thick}(M, \Delta_0)$ in the manifold M be given starting and ending in the $n-1$-face Δ_0, and let $0, 1, ..., n-1$ be its vertices. For any Discrete DG connection with coefficients μ_{ij}^T we define a **Nonabelian Holonomy Representation** along the closed thick path κ with permutation P_κ:

Starting from the initial data $\eta = (\psi_0, ..., \psi_{n-1}) \in R_{\Delta_0}^n$, we step by step calculate the values of function ψ in the vertices of the thick path κ from the equation $Q\psi = 0$. For the thick path $\kappa = (T_N \dots T_1)$ it leads to the linear map

$$\tilde{K}_\kappa : R_{\Delta_0}^n \to R_{\Delta_N}^n = \tilde{K}_{T_N} \dots \tilde{K}_{T_1}$$

where

$$\tilde{K}_T : R_\Delta^n \to R_{\Delta'}^n$$

is the one-step map from the **in-face** into the **out-face** for the n-simplex T provided by the DG connection.

By the **Nonlinear Holonomy Map** we call a product

$$K_\kappa = P_\kappa \tilde{K}_\kappa$$

So the correspondence $\kappa \to K_\kappa$ generates a holonomy representation

$$K : \Omega^{thick}(M, \Delta_0) \to GL_n(k)$$

Lemma 6 *The Holonomy Representation is a Homomorphism*

$$\Omega^{thick}(M, \Delta_0) \to GL_n(k)$$

$$K_{\kappa_2} K_{\kappa_1} = K_{\kappa_2} K_{\kappa_1}$$

For the proof of this lemma we point out that $K = P\tilde{K}$. From the definition of $\tilde{K}$ we have:

$$(P_{\kappa_1}^{-1}\tilde{K}_{\kappa_2}P_{\kappa_1})\tilde{K}_{\kappa_1} = \tilde{K}_{\kappa}$$

because the basis of vertices of the $n-1$-simplex Δ_0 is shifted by the permutation P_{κ_1} after passing the first closed path κ_1. Therefore we obtain finally

$$K_{\kappa_2\kappa_1} = (P_{\kappa_2}P_{\kappa_1})P_{\kappa_1}^{-1}\tilde{K}_{\kappa_2}P_{\kappa_1}\tilde{K}_{\kappa_1} = K_{\kappa_2}K_{\kappa_1}$$

Lemma is proved.

For the unique nontrivial closed thick path in the simplicial star $\kappa \subset ST(\sigma)$ surrounding $n-2$-dimensional simplex σ, this holonomy map reduces to the "Nonabelian Curvature Map" $K_{\sigma,p}$ defined in the previous paragraph where $\Delta = [\sigma, p]$. This simplest path corresponds to the most elementary word $A = a_j^m$ rotating n-simplices around the $n-2$-simplex σ opposite to the vertex $j \in \Delta$.

Lemma 7 *For every closed thick path κ_A determined by the word A and initial $n-1$-simplex Δ_0 the determinant of the Nonabelian Holonomy Map has a form*

$$\det K_{\Delta_0} = (-1)^N \prod_{k=0}^{k=n-1} \mu_k(\kappa_A)$$

where

$$\mu_k(\kappa_A) = \prod_{l=0}^{l=N_k} -\mu_{i_l,i_{l+1}}^{T_l^k}$$

is the product along the "angle" axis R_k going up with the variable x_n from the vertex corresponding to $k \in \Delta_0$ in the abstract model of the thick path. The quantities $\mu_k(\kappa_A)$ are equal to the Abelian Framed Holonomy Representation of the framed paths

$$\gamma_{A,k} = < j_0, ..., j_{N_k} | T_0^k, ..., T_{N_{k-1}}^k >$$

called the "angle" paths of the thick path κ_A

$$\mu_k(\kappa_A) = \mu(\gamma_{A,k})$$

The product of all angle paths is closed.

If all local curvature operators $K_{\sigma,p}$ are equal to the unit matrix for all $n-2$-simplices σ, then the Nonabelian Holonomy depends on the class of thick path in the fundamental group $\pi_1(M)$ only.

Lemma 8 *All matrix elements $\alpha_{ij}(\kappa, \Delta_0)$ of the operators $K = P\tilde{K}$ of the Nonabelian Holonomy transform under the abelian gauge transformations as one-dimensional multiplicative cochains $\alpha_{ij} \to h_i/h_j\alpha_{ij}$ where i, j are the vertices of the initial $n-1$-simplex Δ_0 where*

$$K \to HKH^{-1}, H = diag(h_1, ..., h_{n-1})$$

For the local Nonabelian Curvature Operators we have $P = 1, \Delta_0 = [\sigma, p]$ where σ is an arbitrary $n-2$-simplex, and $p = 1, ..., m$ is a vertex in its simplicial star $p \in ST(\sigma)$. All gauge-invariant polinomial in the variables $\alpha_{ij}(\kappa_A), \mu_{ij}^{TT'}$ can be expressed as polinomials from the the framed abelian holonomy of the closed paths in M.

Proof of this lemmas presents no difficulties. For the proof let us point out that our matrix elements α_{ij} can be expressed as the sums of the products of the quantities $\mu^{T_l}_{i_l, j_l}$ along the paths easily visible as the paths monotonically going up (see Fig) in the abstract model of thick path, starting in the simplex Δ_0 and ending in the upper $n-1$-simplex (who coincides with Δ_0 for the closed paths in the manifold.) Therefore after the gauge transformations only the boundaries of the paths will give contribution, so only the ends i, j remain in the final answer. At the same time, all gauge invariant expressions presented as polynomials of the path integrals of the quantities μ can be expressed through the holonomy of the closed paths (no free ends can be left for the gauge invariant expression). Our lemma is proved.

Example 1 *Let us consider an interesting example of the* **Canonical Connection** *on the triangulated manifolds M where all connection coefficients are equal to one in every simplex $b_{T:i} = 1$. We have also $\mu^T_{ij} = -1$. This connection has been considered in [4]. It appeared also in the work [5] in the different terminology for the needs of the coloring problem. Its image belongs to the group S_{n+1} but we normally realize this group linearly $S_{n+1} \subset GL_n$ using the imbedding $R^n \subset R^{n+1}$ as a subspace invariant under the permutation of all coordinates ψ_i where $\sum \psi_i = 0, i = 0, ..., n-1$. Exactly that corresponds to the canonical connection in the work [4]. Starting from any $n-2$-simplex Δ_0, we construct a* **Coloring of the Vertices by the $n+1$ colors** *$u_0, ..., u_n$ along the thick path with combinatorics corresponding to the word A. Assigning to the initial vertices of the in-simplex Δ_0 the colors $u_0, ..., u_{n-1}$, we can see that the final coloring of the vertices of the out-simplex is uniquely defined* *by two factors: by the combinatorics of the word A and by the realization of this final out-simplex in the manifold M, i.e. by the permutation $P(\kappa)$. We denote a nonabelian holonomy map associated with this connection by the*

$$K^{can}_{\kappa} = P_{\kappa}\tilde{K}^{can}_{\kappa}$$

Lemma 9 *For any closed thick path κ with combinatorics corresponding to the word $A = a^{r_q}_{i_q}...a^{r_0}_{i_0}$ the resulting permutation corresponding to the holonomy of the canonical connection is given by the formula*

$$K^{can}_{\kappa} = P_{\kappa}\tau^{r_q}_{i_q,n}...\tau^{r_0}_{i_0,n}$$

where $i = 0, 1, ..., n-1$ and $\tau_{i,n}$ is a permutation of the points i, n only, $\tau^2_{i,n} = 1$. The permutation P in this formula permutes only the numbers $0, 1, ..., n-1$ leaving the index n invariant.

Proof of this lemma follows immediately from the definition of the abstract model of the thick path.

III. Solution of the Reconstruction Problem for the case $n = 2$
Flat connections. The case $n \geq 2$.

Consider now any oriented trangulated 2-manifold M with the vertices $i, j, ...$, 2-simplices $T, T', ...$ and with the discrete DG Connection. Our field is $k = R, C$ only.

Theorem 2 *All coefficients μ^T_{ij} of the discrete DG Connection over the field k can be recovered up to abelian gauge transformation from the abelian framed holonomy representation*

$$\mu : H^{fr}_1(M) \rightarrow k^*$$

where the framed abelian holonomy image of the group $H^{fr}_1(M)$ is generated by the elements $\rho_{ij} = \mu^T_{ij}\mu^{T'}_{ji} =< i, j, i|TT' >^*$ and by the images of the generators of the ordinary homology group $\mu(\gamma_s), \mu(a_q) \in k^*$.

Proof. Let us describe the reconstruction process. Introduce the new quantities by the following formula:

$$\lambda_{ij} = -\mu^T_{ij}/\sqrt{\rho^{TT'}_{ij}}, \partial T = [ij] + ..., \partial T' = [ji] + ...$$

where some specific value of the square root is chosen. If our manifold is oriented, we choose the orientation of the triangles T, T' to be the same as the global orientation of manifold, and $\partial T = [ij] + \ldots$. In that case we forget about the indices TT' in the formulas, so we have $\lambda_{ij} = -\mu_{ij}/\sqrt{\rho_{ij}} = \sqrt{-\mu_{ij}^{T}/\mu_{ji}^{T'}}$ where

$$\rho_{ij} = \rho_{ji}, \lambda_{ij} = \lambda_{ji}^{-1}$$

Nonuniqueness of choosing the square root we resolve by choosing square roots separately defining $\sqrt{-\mu_{ij}^{T}}$ in every triangle T with requirement

$$\sqrt{-\mu_{ij}}\sqrt{-\mu_{jl}}\sqrt{-\mu_{li}} = 1$$

. We shall return to these details later.

Lemma 10 *For the coboundary of the k^*-valued multiplicative cochain $\lambda = (\lambda_{ij})$ defined by the formula*

$$d\lambda(T) = \lambda(\partial T) = \lambda_{ij}\lambda_{jl}\lambda_{li}, T = [ijl]$$

we have

$$d\lambda(T) = (\rho_{ij}\rho_{jl}\rho_{li})^{-1/2} = \rho^{-1/2}(T)$$

Therefore for every finite triangulated domain D in the manifold M following integral formula is true expressing the integral of the "Curvature" $(\rho(T))^{-1/2}$ along the domain D through the framed abelian holonomy of the boundary curves $\partial D = \bigcup_q \gamma_q$ with framing by the triangles looking in the external direction to the domain D:

$$\prod_{T \in D} (\mu_{ij}^{T})^{-1/2} = (\mu(\partial D))^{-1/2}$$

In particular, for the compact oriented manifold M we have

$$\prod_{T \in M} (\rho(T))^{-1/2} = 1$$

The proof of this lemma follows immediately from definition of the quantities ρ_{ij} and λ_{ij} taking into account the equality $\mu_{ij}\mu_{jl}\mu_{li} = -1$ and our agreement that $\mu_{ij}^{T} = \mu_{ij}$ for the right orientation leading to the conclusion that $\mu_{ji} = \mu_{ji}^{T'}$ where $T' \neq T$, and $\rho_{ij} = \mu_{ij}\mu_{ji} = \rho_{ji}$. Our Lemma is proved.

As a corollary, we are coming to the following conclusion: knowing ρ_{ij} we can reconstruct an unknown cochain λ. After that we define our DG Connection by the formula

$$\mu_{ij} = -\lambda_{ij}\sqrt{\rho_{ij}}$$

By definition, this is a solution of our system. A cochain λ is nonunique: any cocycle δ may be used to change it: $\lambda' = \lambda\delta$, i.e. $\lambda'_{ij} = \lambda_{ij}\delta_{ij}$ where $\delta_{ij}\delta_{jl}\delta_{li} = 1$ for ever triangle $[ijl]$. It is obvious that there is nothing except the set of all 1-cocycles δ and all possible changes of signs in the definition of the square roots of ρ_{ij} what may lead to the same set of the data $\{\rho_{ij}\}$. Making an arbitrary abelian Gauge transformations $\mu_{ij} \to \mu'_{ij} = (h_i/h_j)\mu_{ij}$ we change λ by the cocycle $\delta_{ij} = h_i/h_j$, i.e. by the coboundary. Changing signs of the square roots

$$(\rho_{ij})^{1/2} \to -(\rho_{ij})^{1/2}$$

we change λ by the same signs, so the resulting value of μ remains unchanged.

Changing λ by the cocycle δ nonhomologous to zero, we also change μ by the same δ, i.e. $\mu \to \mu\delta = \mu'$. Therefore our framed abelian holonomy along the closed contours will be changed by the integrals

$$\mu(\gamma) \to \mu(\gamma) \prod_{[ij]\in\gamma} \delta_{ij}$$

Now we are fixing τ by the requirement of the theorem that the framed abelian holonomy is prescribed along some basis γ_k of the homology group $H_1(M)$. Our theorem is proved.

Lemma 11 *Let $\lambda_{ij}^{TT'} = \mu_{ij}^T/\sqrt{\rho_{ij}^{TT'}}$ and $\lambda_{ij}^{TT'} \to \lambda_{ij}^{TT'}\delta_{ij}$, $\mu_{ij}^T \to \mu_{ij}^T\delta_{ij}$ where δ is an ordinary multiplicative cocycle. Then the framed abelian holonomy is changed by the " multiplicative integral" of δ along the same closed paths. The operators of Nonabelian Holonomy along the Thick Paths are changed in the following way:*

$$K_\kappa \to C^{-1}HK_\kappa H^{-1}$$

where $C = \delta_\kappa$ is a value of the 1-cocycle δ on the element $\kappa \in \pi_1(M)$ of the fundamental group, Δ_0 is an initial $n-1$-simplex of the thick path κ and $H = diag(\hat{h}_0, ..., \hat{h}_{n-1})$ is the diagonal matrix whose entries are well-defined up to the common nonzero multiplier, and $\hat{h}_i/\hat{h}_j = \delta_{ij}$.

Proof. The framed abelian part of this lemma is obvious from definitions of λ and δ. In order to prove nonabelian part, we consider this Discrete DG Connection on the special abelian covering $\pi : \hat{M} \to M$ such that our cocycle became exact $\pi^*\delta = d\hat{h}$. Consider any closed thick path κ starting and ending in the $n-1$-simplex Δ_0 of the manifold M and realizing a generator $\gamma \in H_1(M)$. Without any losses of generality, we may think that our cocycle δ has nontrivial "multiplicative integral" along this basic element only, and that our covering is cyclic. On the covering space $\hat{M}$ we choose a covering $n-1$-simplex $\Delta, \pi(\Delta) = \Delta_0$. After that we get a unique covering thick path $\hat{\kappa}$ starting in Δ. This path is a covering path over the closed path κ with period N, i.e. it consists of the sequence of n-simplices $..., T_1, ..., T_{N-1}, T_N, ...$ such that $\pi(T_i) = \pi(T_{N+i})$, and $T_{N-1} \bigcap T_N \in (\pi)^{-1}(\Delta_0)$. The monodromy map on the covering space $R = \hat{\gamma} : \hat{M} \to \hat{M}$ transforms thick covering path into itself, and $R(T_1) = T_N, R(\Delta) = T_{N-1} \bigcap T_N$. Consider the function $\hat{h}$ in the covering thick path. It is nonperiodic: we have $R^*(\hat{h}) = C\hat{h}$ where $C = \prod_{[ij]\in\gamma} \delta_{ij}$ by definition. In the covering path our DG Connections both are periodic and gauge (abelian) equivalent to each other but the equivalence is nonperiodic. According to the lemma 6 (above) we can see that the matrix elements of our Nonabelian Holonomy Operator transform by the following formula

$$\alpha_{ij} \to \hat{h}_i / R^*(\hat{h}_j)\alpha_{ij}, i, j \in \Delta_0$$

because our thick path starts at Δ and ends at $R(\Delta)$ in the covering space. At the same time, we have $R^*(\hat{h}) = C\hat{h}$.This is exactly the statement of our lemma. Lemma is proved.

Theorem 3 *For every data $\rho_{ij}^{TT'}$ on the orientable 2-manifold M corresponding to flat GL_2 connections (i.e. with trivial nonabelian local curvature) , there exist exactly one SL_2-connection up to abelian gauge transformation and changing sign.*

Remark 2 *The existence of SL_n-connection follows from the same arguments also for all $n \geq 2$, but the uniqueness for $n > 2$ will be proved later using some additional arguments not presented yet(see below).*

Proof of this theorem follows from the previous lemma: we may change determinant on the Nonabelian Holonomy Group multiplying ρ_{ij} by the 1-cocycle δ_{ij}. Let us point out that after making determinant equal to 1, we may also change sign of the holonomy.

The first Chern Number: Let now $n = 2$. Consider the case $k = C$ and assume that $|\arg[\rho_{ij}^{TT'}]| < \pi/2$. We define an integer-valued cohomology class $c_1 \in H^2(M, Z)$ by the formula for the cochain:

$$c_1(T) = \frac{1}{2\pi i} \arg[(\rho(T))^{-1/2}]$$

From the equality

$$\prod_T (\rho(T))^{-1/2} = 1$$

for the closed oriented manifold we obtain the result

$$\sum_{T \in M} c_1(T) = r \in Z$$

Our condition permits us to make such unique choice that

$$|\arg[\rho(T)^{-1/2}]| < \pi$$

in all cases. With this agreement we are coming to the well-defined integer number.

IV.Multidimensional Discrete DG Connections.

Consider now any $n > 2$. We expect that all DG Connection can be reconstructed from the framed abelian representation. Let a closed oriented triangulated manifold M be given, and s_k are the numbers of k-simplices. We know a few number of general relations for these numbers:

$$s_{n-1} = \frac{n+1}{2} s_n$$

$$\sum_{k=0}^{k=n} (-1)^k s_k = \eta(M)$$

where $\eta(M)$ is the Euler characteristics. In particular, $\eta = 0$ for the odd dimensions $n = 2t + 1$. Let us present the numbers s_k in the form:

$$s_k = \frac{(n+1)!}{(k+1)!(n-k)!m_k} s_n$$

The meaning of this is following: every n-simplex has exactly $(n+1)!/(k+1)!(n-k)!$ faces of the dimension k. If every k-simplex belongs exactly to m_k n-dimensional simplices, one can deduce that this number s_k can be computed exactly as it is written here. For example, we have always $m_{n-1} = 2$ in the manifolds, $3 \leq m_{n-2} < m_{n-3} < ... < m_0$. In general, this formula gives definition of the numbers m_k as some sort of **the mean value of the number of n-simplices containing a random k-simplex**.

Example 2 *As a simplest example we take sphere* $M = S^n = \partial\Delta_{n+1}$ *as a boundary of the* $(n+1)$*-simplex. In this case* $m_{n-k} = k+1$.

Let us try to count a number of parameters in our reconstruction problem.

Lemma 12 *The number* (μ) *of independent quantities* $\mu_{ij}^{TT'}$ *modulo the abelian gauge transformations is equal to* $ns_n - s_0 + 1 = (\mu)$.

Proof. In every n-simplex $T = [0, 1, ..., n]$ we have exactly n-dimensional manifold of the quantities μ_{ij}^T with (multiplicative) basis μ_{0i}^T according to the relations indicated in their definition above. Different n-simplices are completely independent. Applying the abelian gauge transformations we extract exactly $s_0 - 1$ parameter because the constant function leads to the trivial gauge transformation. This argument implies the statement of our lemma.

For every $n \geq 2$ we define the number (ρ) equal to the (multiplicative) dimension of the set of all quantities $\rho_{ij}^{TT'}$ plus $b = b_{k^*}$ where b is the rank of the Betti number b_1 for the real positive holonomy representation $k^* = R_+$, and it is equal to the larger number $b_1^* = b_1 + torsion_1$ for $k^* = C^*$. We present it by the expression

$$(\rho) = (n-1)s_{n-1} - (n-2)s_{n-2} + b + R$$

where R is the remaining part.

In order to explain meaning of this phrase, let us point out that by the same reason there exist exactly $n-1$-dimensional space of quantities $\rho_{ij}^{TT'}$ in every $n-1$-simplex $T \bigcap T'$. However, they are not independent for different pairs of neighbors T, T'. According to the lemmas above (see local nonabelian curvature) there are $n-1$ different expressions for the quantity $\mu(\sigma) = \det K_{\sigma,p}$ in the star $St(\sigma)$ of every $n-2$-simplex σ depending on the vertex of this simplex σ. It is very probable that the number of remaining relations does not depend on triangulation (at least for $n = 3$. What is important is that these relations are not independent for $n > 2$ in general as we shall see below. Therefore some unknown number R enters our calculation making the answer totally unclear.

Example 3 *Consider the simplest case* $n = 2$ *where everything is already known. The number* (μ) *is described by this formula for all* $n \geq 2$. *For the number* (ρ) *we have* $(\rho) = s_1 - 1 + b$ *where* $b = 2g$ *and* $R = -1$ *because there exist a nontrivial global relation* $\prod_{ij,T,T' \in M} \rho_{ij}^{TT'} = 1$. *Taking into account the relations* $s_2 = 3/2 s_3, \eta(M) = s_2 - s_1 + s_0 = 2 - 2g$, *we are coming to the equality*

$$(\mu) = (\rho)$$

for the closed oriented 2-manifolds. The neighboring oriented pairs TT' *are chosen such that* $\partial T = [ij] + ..., \partial T' = [ji] + ...$ *It corresponds to the fact established above that for the reconstruction of the DG Connection we have to include in the data also values of the framed abelian holonomy on the set of* $2g$ *closed paths presenting the basis of the* $H_1(M)$ *except all* ρ_{ij}.

The case $n \geq 3$

Example 4 *For the case* $n = 3$ *we have* $(\mu) = 3s_3 - s_0 + 1$ *and* $s_2 = 2s_3, s_3 - s_2 + s_1 - s_0 = 0$. *Therefore we obtain as a corollary of the relation* $3s_3 - s_0 + 1 = 2s_2 - s_1 + b + R$ *that*

$$R + b = 1$$

For the homological 3-spheres we have $b = 0$ *and* $R = 1$.

We are going to prove below that the framed abelian holonomy uniquely determines the DG Connection up to the abelian gauge transformation for all $n \geq 3$. Therefore the dimension $\rho^{realizable}$ generated by all $\rho_{ij}^{TT'}$ generated by the DG Connections is equal for $n \geq 3$ to the number $(\mu) - b$. For example, we have for $n = 3$:

$$\rho^{realizable} = 3s_3 - s_0 + 1 - b = 2s_2 - s_1 + R$$

where $R = 1 - b$. The space of all functions with formal properties of the quantities like $\rho_{ij}^{TT'}$ and general local relations indicated above for $n = 3$ has dimension equal to $2s_2 - s_1$ plus something depending on the topology of the manifold M only. Looking on the right-hand side of this relation, we expect to find out **exactly one global dependence** between the s_1 already known general "local" relations for these quantities in the simplicial stars of all $n-2$-simplices, **plus exactly** b **"global" relations** depending on the 1-cycles in the n-manifold M as a minimal possibility. We shall describe these relations below for the closed oriented 3-manifolds. Let now $n \geq 3$.

Theorem 4 *Let $k^* = C^*$.*

I. For every integral 2-cycle $z = \sum \Delta_k, a_j \in Z$ presented as a sum of the oriented 2-simplices in the n -manifold M there is a relation between the quantities $\rho_{ij}^{TT'} \in C^*$: Let an arbitrary "framing" be chosen along this 2-cycle, i.e. with every 2-simplex $\Delta_k \subset z$ we associate an n-simplex T such that $\partial T = \Delta_k + \dots$. This relation has a form

$$\prod_{[ij] \in \Delta_k \bigcap \Delta_{k'} \subset z} \rho_{ij}^{T_k T_{k'}} = 1$$

II.For every 1-dimensional "torsion cycle" $a \in H_1(M, Z)$ of the order r (i.e. $a^r = 1$ in the multiplicative form) there is a relation between the abelian framed holonomy $\mu(a)$ along the framed path a and the quantities $\rho_{ij}^{TT'}$. Let an integral chain (simply, a formal sum of the oriented 2-simplices) $u = \sum_k \Delta_k$ with some framing T_k, $\partial T_k = \Delta_k + \dots$ is given such that $\partial u = a^r$. The relation has a form

$$\prod_{[ij] \in \Delta_k \bigcap \Delta_{k'} \subset u} \rho_{ij}^{T_k T_{k'}} = \mu(\partial u) = \mu(a)^r$$

, and (∂u) is a boundary 1-chain with induced framing. The product along the chain u is defined representing our chain u through the pairs of 2-simplices $\Delta_k, \Delta_{k'}$ whose intersection is exactly the edge $[ij]$ entering them with the opposite orientations.

Corollary 3 *For the closed oriented 3-manifolds M we have $b_2 = b_1$ by the Poincare duality. Therefore we have $b = b_2 + torsion_1$, and the number of relations is equal to the rank b of the topological part of the framed abelian holonomy.*

The proof of this theorem follows from the integral formula:

Lemma 13 *For every integral 2-chain $w = \sum_k \Delta_k$ with given framing of the 2-simplices following "Integral Formula" is true:*

$$\prod_{[ij]\in \Delta_k \bigcap \Delta_{k'} \subset w} \rho_{ij}^{TT'} = \prod_{[ij]\in \partial w} -\mu_{ij}^{T}$$

The first Chern class

As a by-product of this theorem we define a first Chern class

$$c_1 \in H^2(M, Z)$$

in the same way as for $n = 2$ above: we put

$$(c_1, z) = (2\pi)^{-1} \sum_{[ij]\in \Delta_k \bigcap \Delta_{k'}} \arg[\rho_{ij}^{T_k T_{k'}}] \in Z$$

for every cycle z; For the cycles modulo finite order we define

$$(c_1, u) = (2\pi)^{-1} \sum \arg \rho_{ij}^{T_k T_{k'}} \in Z/rZ$$

as in the theorem above. The topological properties of these quantities in the discrete case should be especially discussed. We avoid this discussion here.

Reconstruction of the Connection for $n \geq 3$

Let us start with the case $n = 3$. Consider now a three-dimensional oriented manifold M. For every simlicial star $St(\sigma)$ of one dimensional simplex $\sigma = (ij) \subset M$ we reconstruct the quantities μ_{ij}^T up to the unknown constant $\delta_{ij}, T \in St_\sigma$, from the equations

$$\rho_{ij}^{T_p T_{p-1}} = \mu_{ij}^{T_p} \mu_{ji}^{T_{p-1}}, p = 1, 2, \ldots q(\sigma)$$

These equations are solvable because the trivial set of relations B is satisfied (see Lemma 1). Here the simplices $T_p \in St(\sigma)$ are numerated in the natural cyclic order modulo $q(\sigma)$ where $T_{q+1} = T_1$. So we know the quantities

$$\tilde{\mu}_{ij}^{T_p} = \mu_{ij}^{T_p} \delta_{ij}$$

where $\delta_{ij}\delta_{ji} = 1$, if solution $\mu^T_{ij} = (\mu^T_{ji})^{-1}$ exists. Let us consider the necessary equation

$$\mu^T_{ij}\mu^T_{jl}\mu^T_{li} = -1$$

We are coming to the conclusion that our problem is solvable if and only if following three requirements are satisfied: 1.The quantity

$$\tilde{\mu}^T_{ij}\tilde{\mu}^T_{jl}\tilde{\mu}^T_{li} = \tilde{\mu}[\Delta]^T$$

is in fact cochain depending only on the oriented simplex $[ijl] = \Delta$, i.e. $\tilde{\mu}[ijl]\tilde{\mu}[jil]] = 1$; 2. This cochain is closed; 3.This cochain is exact.

Proof of the Statement 1: We define this quantity on the oriented manifolds M for the 3-simplices T of the right orientation such that $\partial T = [ijl] + \ldots$. This agreement makes it well-defined as a function of oriented simplices. In order to prove that $\tilde{\mu}[ijl] = \tilde{\mu}[jil]^{-1}$, we use the identity

$$\rho^{TT'}_{ij}\rho^{TT'}_{jl}\rho^{TT'}_{li} = 1$$

for the pair of oriented simplices such that $\Delta = T\bigcap T'$. We know that $\rho^{TT'}_{ij} = \tilde{\mu}^T_{ij}\tilde{\mu}^{T'}_{ji}$. This equality immediately implies our result. The statement 1 is proved.

Proof of the Statement 2. For every 3-simplex T we consider a full product of the quantities $\tilde{\mu}[\Delta_s]$ along the boundary simplices $\Delta_s, s = 1,2,3,4$. It is easy to see that for every edge $ij \subset T$ we have exactly two multipliers in this product equal to $\tilde{\mu}^T_{ij}$ and $\tilde{\mu}^T_{ji}$. We use here the result of the statement 1 expressing everything through the quantities $\tilde{\mu}^T$ for the faces of any orientations (other 3-simplices are not needed). This property implies our result. The statement 2 is proved.

Proof of the Statement 3. We know already that this is a multiplicative cocycle with values in C^*. As everybody knows, its exactness requires the conditions formulated in the Theorem above where the homological relations for the connection coefficients were found. Therefore the statement 3 follows from the homological arguments.

Using this result, we easily reconstruct our connection μ^T_{ij}: we take any solution to the equation

$$d\delta = \tilde{\mu}[\Delta]$$

and put $\mu^T_{ij} = \delta_{ji}\tilde{\mu}^T_{ij}$. This solution satisfies to all requirements and define exactly the same local part of the framed abelian holonomy representation.

This set of quantities can be choose modulo closed 1-cocycle $\delta_{ij} \to \delta'_{ij}$ such that their ratio is closed. This degree of freedom should be used in order to make the proper adjustment of the global part of framed abelian holonomy along the basis of the first homology group. The exact part of this cocycle is responsible for the abelian gauge transformations. These arguments finish our problem.

Nonorientable case can be easily reduced to the oriented one using the orientable 2-covering in the same way as for the case $n = 2$.

For the manifolds M of all dimensions $n \geq 3$ we develop the same reconstruction process as for $n = 3$.

Step 1. Consider the simplicial star $St(ij)$ of the edge ij. Solve the equations

$$\rho_{ij}^{TT'} = \tilde{\mu}_{ij}^{T} \tilde{\mu}_{ji}^{T'}$$

in the star. This problem can be solved uniquely up to unknown constant δ:

$$\mu_{ij}^{T} = \delta_{ij} \tilde{\mu}_{ij}^{T}, \delta_{ij}\delta_{ji} = 1$$

This is true because $\rho_{ij}^{TT'}$ is a 1-cocycle in the dual cell decomposition of the star $St(ij)$ where T are the vertices (see the relations B in Lemma 1). So our condition for the solvability of that intermediate problem is $H^1(St(ij), k^*) = 1$. It is certainly true in the manifolds.

Step 2. In order to solve the equation $\mu_{ij}^{T}\mu_{jl}^{T}\mu_{li}^{T} = -1$ for every n-simplex T we need to prove that this quantity

$$\tilde{\mu}_{ij}^{T}\tilde{\mu}_{jl}^{T}\tilde{\mu}_{li}^{T} = \tilde{\mu}^{T}[ijl]$$

is in fact a well-defined multiplicative cocycle independent on T. This statement follows from the requirement

$$\rho_{ij}^{TT'}\rho_{jl}^{TT'}\rho_{li}^{TT'} = 1$$

where $\rho_{ij}^{TT'} = \tilde{\mu}_{ij}^{T}\tilde{\mu}_{ji}^{T'}$. So we conclude that $\tilde{\mu}^{T}[ijl]\tilde{\mu}^{T'}[jil] = 1$ for every pair T, T'. The proof that this 2-cochain is closed is the same as for the case $n = 3$.

Step 3. In order to prove that this cochain is exact we use an analog of the same relations for the quantities ρ integrated along the cycles (see the Theorem above). This theorem gives us the set of relations which leads to the property of any cocycle to be exact in the elementary homological algebra.

After that all arguments coincide with the case $n = 3$. Our reconstruction process is finished.

In particular, we see that the Uniqueness Theorem for all $n \geq 2$ follows from our results:

Theorem 5 *The framed abelian holonomy representation determines completely the Discrete* GL_n*-Connection* $\{\mu_{ij}^T\}$ *on the triangulated* n*-manifold* M *up to the abelian gauge transformation. The set of conditions on the data of the framed abelian holonomy representation found in this work is necessary and sufficient for the reconstruction.*

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Dedicated to V. I. Arnol'd

Geometry of the Triangle Equation on Two-Manifolds*

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Abstract

A non-traditional approach to the discretization of differential-geometrical connections was suggested by the authors in 1997. At the same time we started studying first order difference "black and white triangle operators (equations)" on triangulated surfaces with black and white coloring of triangles. In this work, we develop the theory of these operators and equations, showing their similarity with the complex derivatives ∂ and $\bar{\partial}$.

Introduction

Do there exist any natural difference analogs of the operators ∂ and $\bar{\partial}$ on the complex plane? No theory of difference "first order" operators that have some properties similar to those of ∂ and $\bar{\partial}$ has been developed before; the attention of the literature is paid only to discrete analogs of the Laplace–Beltrami operator, which are "second order" operators. However, in papers [1, 2, 3] we already studied some "first order" difference operators, starting

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from the problem of finding a discrete analog of the Laplace transformation for a standard two-dimensional second order linear differential equation. Our operators were originally defined on the equilateral triangular lattice in the space $\mathbb{R}^2$ (see [3]). In [1, 2] these definitions were extended to arbitrary "black and white" triangulations of two-dimensional surfaces without boundary. The "black (white) triangle" equations, which have naturally appeared in the theory, are studied here with respect to their analogy with the operators ∂ and $\bar{\partial}$. It turns out that their properties partially imitate some properties of complex derivatives on two-dimensional manifolds.

1 First order triangle operators on triangulated surfaces. Discrete connections

Let M be a two-dimensional manifolds without boundary endowed with a triangulation $\mathcal{T}$. Let $\mathcal{K}$ be a family of triangles from $\mathcal{T}$ such that, for any vertex P of $\mathcal{T}$, there is at least one triangle $T \in \mathcal{K}$ adjacent to P. Let us fix some numeric coefficients $b_{T,P}$ defined for any triangle $T \in \mathcal{K}$ and its vertex $P \in T$.

Definition 1. By a *first order (or triangle) operator* on M associated with the family $\mathcal{K}$ and coefficients $b_{T,P}$ we call an operator $Q^{\mathcal{K}}$ defined by the formula

$$(Q^{\mathcal{K}}\psi)_T = \sum_{P \in T} b_{T,P}\psi_P. \tag{1}$$

The operator $Q^{\mathcal{K}}$ is well-defined as a linear map of the space of all functions ψ_P of a vertex P into the space of functions $\varphi_T = (Q^{\mathcal{K}}\psi)_T$ of a triangle T from the selected family $\mathcal{K}$.

Remark 1. In 1997 (see [1]) we considered operators on multi-dimensional manifolds as well, but in this paper we focus only on the two-dimensional case, where an analogy with the operators ∂, $\bar{\partial}$ takes place.

The maximal family: discrete connections. In the case of the maximal family $\mathcal{K}$ equal to the set of all triangles of the triangulation $\mathcal{T}$, the theory of the equation

$$Q^{\mathcal{K}}\psi = 0 \tag{2}$$

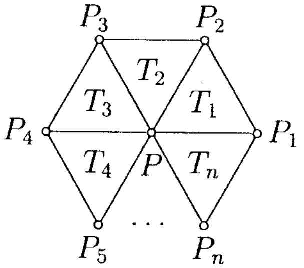
naturally leads to the notions of a *discrete connection* and *curvature*. The coefficients $b_{T,P}$ play the role of the connection coefficients. Following [1], we now introduce the *local holonomy* (or the *curvature*) of the connection $\{b_{T,P}\}$ at a vertex P, which is an operator $\mathbb{R}^2 \to \mathbb{R}^2$ defined by a matrix of the form

$$K_P = \begin{pmatrix} 1 & k'_P \\ 0 & k''_P \end{pmatrix}. \tag{3}$$

It is constructed as follows.

Let us enumerate all the vertices and triangles included in the star of a vertex P as shown in Fig. 1. We will be constructing a local solution of equation (2) in the star of P, starting from arbitrarily chosen values of ψ at vertices P, P_1. We will have

Figure 1:



$$\psi_{P_2} = -\frac{1}{b_{T_1,P_2}}(b_{T_1,P}\psi_P + b_{T_1,P_1}\psi_{P_1}),$$
$$\psi_{P_3} = -\frac{1}{b_{T_2,P_3}}(b_{T_2,P}\psi_P + b_{T_2,P_2}\psi_{P_2}),$$
$$\dots$$
$$\widetilde{\psi}_{P_1} = -\frac{1}{b_{T_n,P_1}}(b_{T_n,P}\psi_P + b_{T_n,P_n}\psi_{P_n}) = k'_P\psi_P + k''_P\psi_{P_1},$$

where

$$k'_P = \sum_{k=0}^{n-1}(-1)^{k+1}\,\frac{b_{T_n,P}\cdot\ldots\cdot b_{T_{n-k},P}}{b_{T_n,P_1}\cdot\ldots\cdot b_{T_{n-k},P_{n-k+1}}}$$
$$k''_P = (-1)^n\,\frac{b_{T_1,P_1}b_{T_2,P_2}\cdot\ldots\cdot b_{T_n,P_n}}{b_{T_1,P_2}b_{T_2,P_3}\cdot\ldots\cdot b_{T_n,P_1}}.$$

(In the formula for k'_P, we assume $P_{n+1} = P_1$.)

Thus, having started from arbitrary values ψ_P, ψ_{P_1}, after a full turn around the vertex P, we come to new ones

$$(\widetilde{\psi}_P, \widetilde{\psi}_{P_1}) = (\psi_P, \psi_{P_1}) \begin{pmatrix} 1 & k'_P \\ 0 & k''_P \end{pmatrix}.$$

We say that the connection $\{b_{T,P}\}$ has *zero curvature* at the vertex P if, for any ψ_P, ψ'_P, we have $(\widetilde{\psi}_P, \widetilde{\psi}_{P_1}) = (\psi_P, \psi_{P_1})$, *i.e.*, if we have

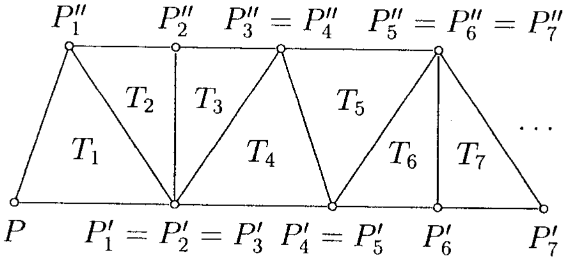
$$k'_P = 0, \quad k''_P = 1.$$

Example 1. Assume that all the connection coefficients are equal to one, $b_{T,P} \equiv 1$, and the number n_P of triangles adjacent to a vertex P is even, $n_P = 2l_P$. Then the holonomy of the connection $\{b_{T,P}\}$ at the vertex P is trivial. Indeed, we have

$$(\psi_P, \psi_{P_{k+1}}) = (\psi_P, \psi_{P_k}) \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad K_P = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}^{n_P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The *global holonomy* of a discrete connection $\{b_{T,P}\}$ is defined for any "thick" path γ, which is a sequence of triangles $T_1, \dots, T_m$ such that, for all $j = 1, \dots, m-1$, the triangles T_j and T_{j+1} have a common edge $\kappa_j = T_j \cap T_{j+1}$, whose vertices are denoted by P'_j, P''_j (see Fig. 2). This is done as follows.

Figure 2: A thick path



Start from a solution $(\psi_P, \psi_{P'_1}, \psi_{P''_1})$ of equation (2) in the triangle T_1:

$$b_{T_1,P}\psi_P + b_{T_1,P'_1}\psi_{P'_1} + b_{T_1,P''_1} = 0.$$

We assume that ψ_P and $\psi_{P'_1}$ are chosen arbitrarily, and $\psi_{P''_1}$ is found from the equation. Then we extend the solution along γ successively to the triangles

$T_2, T_3, \dots$, *i.e.*, to the vertices $P_2', P_2'', P_3', P_3'', \dots$. This can be done in a unique way. If γ is a loop, *i.e.*, $T_{m+1} = T_1$, then, having passed over γ, we will obtain a new solution $(\widetilde{\psi}_P, \widetilde{\psi}_{P_1'}, \widetilde{\psi}_{P_1''})$ of (2) in the triangle T_1. Thus, we have constructed the *holonomy operator* $R_\gamma : \mathbb{R}^2 \to \mathbb{R}^2$:

$$(\psi_P, \psi_{P_1'}) \mapsto (\widetilde{\psi}_P, \widetilde{\psi}_{P_1'}) = (\psi_P, \psi_{P_1'}) \cdot R_\gamma.$$

Lemma 1. *The global holonomy of any discrete connection $\{b_{P,T}\}$ with zero curvature is a well defined homomorphism $\pi_1(M, T_1) \to GL(2, \mathbb{R})$.*

Proof. For any two adjacent triangles T, T', *i.e.*, ones having exactly one common edge, we denote by $R_{T,T'}$ the extension operator of a solution of (2) from the triangle T to T': $\psi^{(T)} \mapsto \psi^{(T')} = \psi^{(T)} \cdot R_{T,T'}$. By definition, we have

$$R_\gamma = R_{T_1,T_2} \cdot \ldots \cdot R_{T_{m-1},T_m} \cdot R_{T_m,T_{m+1}}.$$

Clearly, the correspondence $\gamma \mapsto R_\gamma$ is multiplicative:

$$R_{\gamma_1 \circ \gamma_2} = R_{\gamma_1} \circ R_{\gamma_2},$$

and consistent with the inversion:

$$R_{\gamma^{-1}} = (R_\gamma)^{-1}.$$

We have only to check the invariance of R_γ under a homotopy of the path γ.

Any homotopy of γ can be reduced to finitely many insertions and removals of fragments like this:

$$\begin{aligned} \dots, T, T', T, \dots \quad &\leftrightarrow \quad \dots, T, \dots, \\ \dots, T, T', T'', \dots, T^{(n)}, T, \dots \quad &\leftrightarrow \quad \dots, T, \dots, \end{aligned}$$

where $T, T', \dots, T^{(n)}$ are triangles from the star of a vertex listed in the order we meet them when walking around the vertex. The holonomy operator R_γ is unchanged under the first operation in view of $R_{T,T'} R_{T',T} = \mathrm{id}$, and it does so under the latter in view of the triviality of the local holonomy. □

Theorem 1. *For any triangulation $\mathcal{T}$ of a surface M and any homomorphism $\rho : \pi_1(M) \to GL(2, \mathbb{R})$ there exists a zero curvature discrete connection $\{b_{T,P}\}$ that generates the representation ρ.*

See Appendix for a proof of this theorem.

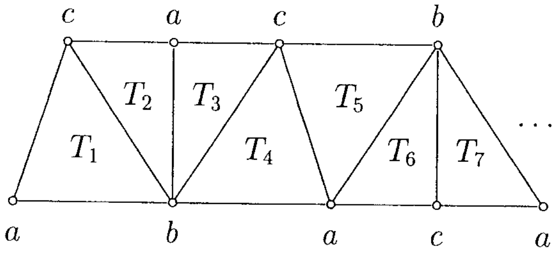
In the sequel, we will consider in detail only one special case in which the star of each vertex of $\mathcal{T}$ consists of an even number of triangles and all the connection coefficients are equal to 1, $b_{T,P} \equiv 1$. As we have already seen, in this case, the connection $\{b_{T,P}\}$ has zero curvature, and hence, defines a homomorphism $\rho : \pi_1(M) \to GL(2, \mathbb{R})$. We call this connection *canonical.*

Proposition 1. *The holonomy group of the canonical connection associated with a triangulation all whose vertices have an even valence is a subgroup of the group $S_3 \subset GL(2, \mathbb{R})$ generated by matrices*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{4}$$

Proof. Let γ be a thick path $T_1, T_2, \ldots, T_{m+1} = T_1$. Let us color the vertices of T_1 into three distinct colors, say, a, b, c, and then continue the coloring so that all three vertices of any triangle T_j, $j = 2, 3, \ldots$, have different colors (see Fig. 3). At the same time, we extend a solution ψ from the triangle T_1 to $T_2, T_3, \ldots$. Since the sum of the values of ψ over the vertices of any triangle must be zero, we will have $\psi_P = \psi_{P'}$ for any two vertices P, P' of the same color. We denote by ψ_a, ψ_b, ψ_c the values of ψ at vertices of the corresponding color. We have $\psi_a + \psi_b + \psi_c = 0$.

Figure 3: Coloring vertices in three colors



After passing over γ, we come to another coloring of the vertices of T_1, which is obtained from the original one by a permutation from S_3. The group S_3 is generated by the permutations $\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$ and $\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$. For the first permutation, we have

$$(\widetilde{\psi}_a, \widetilde{\psi}_b) = (\psi_b, \psi_a) = (\psi_a, \psi_b) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

for the latter one, we have

$$(\widetilde{\psi}_a, \widetilde{\psi}_b) = (-\psi_a - \psi_b, \psi_b) = (\psi_a, \psi_b) \cdot \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

□

In the case of $\{b_{T,P}\}$ the canonical connection, we call equation (2) the *triangle equation*:

$$(Q\psi)_T = \sum_{P \in T} \psi_P = 0. \tag{5}$$

In order to construct a solution, we have to take initial values ψ_a, ψ_b invariant under all holonomy operators. According to Proposition 1, there are three cases:

Case 1 : the holonomy group G is isomorphic either to S_3 or S_3^+ (the group of even permutations); the representation $G \to GL(2, \mathbb{R})$ is irreducible;

Case 2 : the holonomy group is isomorphic to $\mathbb{Z}_2$; up to multiplicative constant, there is exactly one invariant vector;

Case 3 : the holonomy group is trivial; all vectors from $\mathbb{R}^2$ are invariant.

Thus, we have proved the following.

Theorem 2. *The space of "covariant constants", i.e., of solutions of the triangle equation, is:*

two-dimensional if the holonomy group is trivial;

one-dimensional if the holonomy group is isomorphic to $\mathbb{Z}_2$*;*

trivial in the other cases.

Now consider the operator $L = Q^+ Q$, where Q is the first order operator associated with the canonical connection on the surface M. The operator L can be written as follows:

$$L = -2\delta d + 3n_P = -2\Delta + 3n_P, \tag{6}$$

where n_P is the "potential" equal to the valence of a vertex, d, δ are the boundary and coboundary operators, respectively, $d = \delta^+$.

Proposition 2. *The operator L has a non-trivial space of zero modes $L\psi = 0$ if and only if the holonomy group of the canonical connection is either trivial, in which case the space of zero modes is two-dimensional, or isomorphic to $\mathbb{Z}_2$, in which case it is one-dimensional. These modes ψ satisfy the equation $Q\psi = 0$.*

Remark 2. For $\mathcal{T}$ a *uniform* triangulation, *i.e.*, such that all vertices have the same valence $n_P = 2l = \text{const}$, the zero modes of L are eigenfunctions of the Laplace–Beltrami operator Δ with maximal eigenvalue.

It is interesting to compare our constructions with some results previously obtained in graph theory (see [4]). Let us consider the Poincare dual graph Γ of our triangulation. Vertices of Γ correspond to triangles of $\mathcal{T}$. This graph Γ is “dichromatic” in the graph-theoretical terminology if and only if there exists a coloring of triangles of $\mathcal{T}$ in black and white such that any two adjacent triangles have different colors. The Laplace–Beltrami operators on dichromatic graphs Γ were studied in [4]. The space $\mathcal{L}_\Gamma$ of functions depending on vertices of Γ splits into the direct sum

$$\mathcal{L}_\Gamma = \mathcal{L}_{\mathrm{b}} \oplus \mathcal{L}_{\mathrm{w}},$$

where $\mathcal{L}_{\mathrm{b}}$ (respectively, $\mathcal{L}_{\mathrm{w}}$) is the space of functions of black (respectively, white) triangles of $\mathcal{T}$.

In [1] we studied operators $Q_{\mathrm{bw}} : \mathcal{L}_{\mathrm{w}} \to \mathcal{L}_{\mathrm{b}}$, and the operators $Q_{\mathrm{wb}} = Q^+_{\mathrm{bw}}$ adjoint to them, associated with the coloring. They were needed for a generalization of the Laplace transformation. This was a new type of first order difference operators, except in the following case: for an equilateral triangular lattice with the natural coloring, both spaces $\mathcal{L}_{\mathrm{w}}, \mathcal{L}_{\mathrm{b}}$ are canonically isomorphic to the space of functions depending on vertices (see below). In our previous work [1], “second order Schrödinger operators” of the form $L = Q_{\mathrm{wb}}Q^+_{\mathrm{wb}}$ and $L' = Q^+_{\mathrm{wb}}Q_{\mathrm{wb}}$ acting on the spaces $\mathcal{L}_{\mathrm{w}}$ and $\mathcal{L}_{\mathrm{b}}$, respectively, were discussed.

It is easy to see that the standard graph-theoretical Laplace–Beltrami operator on the dual graph Γ has a matrix of the form:

$$\Delta_\Gamma = \begin{pmatrix} 0 & Q_{\mathrm{wb}} \\ Q^+_{\mathrm{wb}} & 0 \end{pmatrix}, \qquad \Delta^2_\Gamma = \begin{pmatrix} Q_{\mathrm{wb}}Q^+_{\mathrm{wb}} & 0 \\ 0 & Q^+_{\mathrm{wb}}Q_{\mathrm{wb}} \end{pmatrix} = L \oplus L'.$$

So the results obtained in the work of P. Sarnak [4] involve, in fact, the operators of the type Q_{wb} but not of the type studied in the present work.

Let us ask the following

Question: when a triangulation $\mathcal{T}$ of a surface M admits a black and white (or b/w) coloring of triangles such that any two adjacent triangles are of different colors?

The answer, which is obvious, is given in the following Lemma.

Lemma 2. *A b/w coloring of a triangulation $\mathcal{T}$ exists if and only if any thick loop $\gamma = (T_1, \dots, T_m, T_{m+1} = T_1)$, where $T_i \cap T_{i+1}$ is an edge for all $i = 1, \dots, m$, has an even length: $m = 2k$.*

Thus, we have the following three homomorphisms $\rho_1, \rho_2, \rho_3 : \pi_1(M) \to \mathbb{Z}_2$:

1) the parity ρ_1 of the global holonomy, $\pi_1(M) \to S_3 \to \mathbb{Z}_2$;
2) the orientation homomorphism ρ_2;
3) the parity homomorphism ρ_3 of the number of triangles in a thick loop.

Lemma 3. *We have $\rho_1\rho_2 = \rho_3$.*

We skip the easy proof.

Corollary 1. *a) Let M be orientable; then a b/w coloring of $\mathcal{T}$ exists if and only if the holonomy group of the canonical connection associated with $\mathcal{T}$ is either trivial or isomorphic to the group S_3^+ of even permutations;*

b) Let M be non-orientable; then a b/w coloring exists if and only if the holonomy group is isomorphic to $\mathbb{Z}_2$ and we have $\rho_1 = \rho_2$.

In the case $\rho_3 = 1$, we fix a b/w coloring of $\mathcal{T}$ and consider the first order operators Q_{b} and Q_{w} of the form (1) associated with the families of black and white triangles, respectively. The corresponding equations $Q_{\mathrm{b}}\psi = 0$ and $Q_{\mathrm{w}}\psi = 0$ are called *black triangle equation* and *white triangle equation*, respectively. Clearly, we have

$$Q_{\mathrm{b}}^+ Q_{\mathrm{b}} = -\delta d + \frac{3}{2} n_P = Q_{\mathrm{w}}^+ Q_{\mathrm{w}}. \tag{7}$$

2 Maximum principle

As before, let M be a surface endowed with a triangulation $\mathcal{T}$. Let $M' \subset M$ be a surface (with boundary) that has a form of a finite simplicial subcomplex of M. Assume that M' is orientable and the holonomy of the canonical connection of M' is globally trivial.

As we saw in the previous Section, there exists a b/w coloring of triangles from M' such that any two adjacent triangles have different colors. Also, there exists a coloring of vertices into three colors, a, b, c, such that all three vertices of any triangle are of different colors. The space V of covariant constants is two-dimensional and consists of functions ψ_P that depend only on the color of a vertex P, and the values ψ_a, ψ_b, ψ_c of ψ at vertices of the corresponding color satisfy the relation $\psi_a + \psi_b + \psi_c = 0$.

Let $\mathcal{K}$ be the set of black triangles in M'. For any solution ψ of the black triangle equation $Q^{\mathcal{K}}\psi = 0$, we define a mapping $\widehat{\psi} : \mathcal{K} \to V \cong \mathbb{R}^2$ from the set of black triangles to the space of covariant constants on M' as follows. For a black triangle $T \subset M'$, we define $\widehat{\psi}(T)$ to be the covariant constant that coincides with ψ on vertices of T. In other words, $\widehat{\psi}(T) = (\psi_{P_a(T)}, \psi_{P_b(T)}, \psi_{P_c(T)})$, where $P_a(T), P_b(T), P_c(T)$ are vertices of T of colors a, b, c, respectively.

Let us denote by $\partial_- M'$ the set of triangles in M' that are adjacent to the boundary $\partial M'$, *i.e.*, such that $T \subset M'$ and $T \cap \partial M' \neq \varnothing$.

Theorem 3 (Maximum principle). *For any solution ψ of the black triangle equation on a submanifold $M' \subset M$, the image $\widehat{\psi}(\mathcal{K})$ of the set of black triangles in the space of covariant constants is contained in the convex hull of the image of the boundary $\widehat{\psi}(\partial_- M')$.*

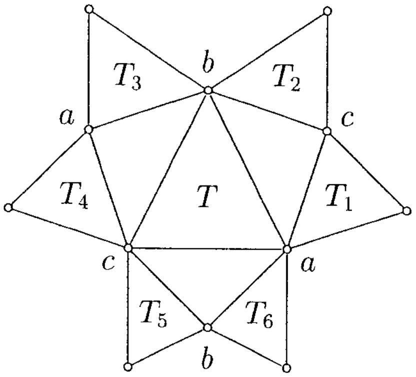
Proof. We assume that ψ is not a covariant constant, in which case the assertion is trivial because the image $\widehat{\psi}(\mathcal{K})$ is just one point $\psi \in V$.

Since the complex M' is finite, the convex hull C of $\widehat{\psi}(\mathcal{K})$ is a polygon in the plane $V \cong \mathbb{R}^2$. The assertion means that the corners of C are the images of boundary triangles $T \in \mathcal{K} \cap (\partial_- M')$. In order to prove this, we are going to show that the image of any internal black triangle, *i.e.*, a triangle $T \in \mathcal{K} \setminus (\partial_- M')$, is not a corner of C.

For an internal black triangle $T \subset M'$, let us denote by $T_1, \ldots, T_6$ the six black triangles that are at distance 2 from T in the sense of thick paths and are arranged as shown in Fig. 4. (The pairs of triangles (T_1, T_6), (T_2, T_3), (T_4, T_5) may coincide, which will not affect the proof.) If T is an internal triangle of M', then all six triangles $T_1, \ldots, T_6$ lie in M'. Without loss of generality we may assume that vertices are colored as showing in Fig. 4.

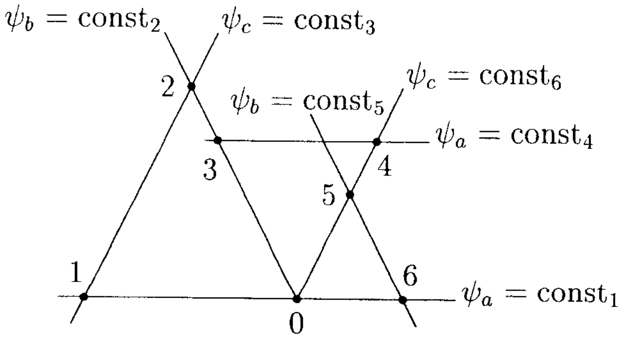
There are three families of parallel straight lines in the space of covariant constants: $\psi_a = \text{const}$, $\psi_b = \text{const}$, and $\psi_c = \text{const}$. If two black triangles

Figure 4:



$T, T' \subset M'$ have a common vertex of color $x \in \{a, b, c\}$, then the points $\widehat{\psi}(T), \widehat{\psi}(T') \in V$ lie in a straight line $\psi_x = \text{const}$. In Fig. 5, one of possible arrangements of points $\widehat{\psi}(T), \widehat{\psi}(T_1), \dots, \widehat{\psi}(T_6)$, which are marked simply as $0, 1, 2, 3, 4, 5, 6$, respectively, is shown. It is easy to see that, for any other possible arrangement, the point $\widehat{\psi}(T)$ lies between either $\widehat{\psi}(T_1), \widehat{\psi}(T_6)$, or $\widehat{\psi}(T_2), \widehat{\psi}(T_3)$, or $\widehat{\psi}(T_4), \widehat{\psi}(T_5)$. $\square$

Figure 5:



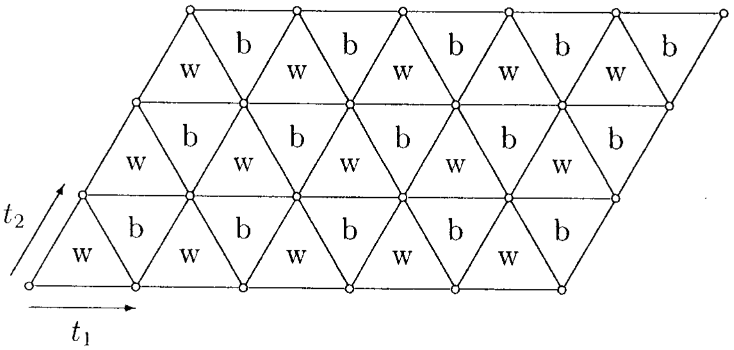
In the particular case $M' = M$, we come to the following

Corollary 2. *Let M be a closed triangulated surface such that the holonomy group of the canonical connection is trivial. Then the only solutions of the black (or white) triangle equation on M are covariant constants, which form a two-dimensional space.*

3 Triangle equations in the Euclidean geometry

From now on, we will consider the case when our surface M is the Euclidean plane $\mathbb{R}^2$ triangulated so that all triangles are equilateral. Its vertices form a two-dimensional lattice $\ell \subset \mathbb{R}^2$, $\ell \cong \mathbb{Z}^2$. We denote by t_1 and t_2 the basis shifts of the lattice. Each triangle of this triangulation is either of the form $\langle P, t_1 \cdot P, t_2 \cdot P \rangle$ (in which case we call it *white* and denote by T_P^{w}), or of the form $\langle P, t_1^{-1} \cdot P, t_2^{-1} \cdot P \rangle$ (in which case we call it *black* and denote by T_P^{b}), see Fig. 6.

Figure 6: Black and white coloring of the Euclidean plane



Thus, in this case, there is a natural one-to-one correspondence between vertices of the triangulation and black (white) triangles: $P \leftrightarrow T_P^{\mathrm{b}}$ (respectively, $P \leftrightarrow T_P^{\mathrm{w}}$). Due to this correspondence, we can think of operators $Q_{\mathrm{b}}, Q_{\mathrm{w}}$ of the form (1) associated with the set of black or white triangles, respectively, as ones mapping the space of functions on the lattice $\mathbb{Z}^2$ to itself. In other words, $Q_{\mathrm{b}}, Q_{\mathrm{w}}$ have the form

$$\begin{aligned} Q_{\mathrm{w}} &= x_n + y_n t_1 + z_n t_2, \\ Q_{\mathrm{b}} &= u_n + v_n t_1^{-1} + w_n t_2^{-1}, \end{aligned} \tag{8}$$

where $n = (n_1, n_2) \in \mathbb{Z}^2$. Notice, that operators of "black" type Q_{b} are (formally) conjugated to operators of "white" type Q_{w}, as we have $t_j^+ = t_j^{-1}$.

Consider a real self-adjoint second order (Schrödinger) operator

$$(L\psi)_P = \sum_{P'} b_{P,P'} \psi_{P'}, \quad b_{P,P'} = b_{P',P}, \tag{9}$$

where $b_{P,P'} > 0$ holds if either we have $P = P'$ or $\langle P, P' \rangle$ is an edge, and we have $b_{P,P'} = 0$ otherwise. In other words, L is an operator of the form

$$L = a_n + b_n t_1 + c_n t_2 + d_n t_1^{-1} t_2 + e_n t_1^{-1} + f_n t_2^{-1} + g_n t_1 t_2^{-1}. \qquad (10)$$

It admits a unique factorization of the form

$$L = Q_{\mathrm{b}}^{+} Q_{\mathrm{b}} + U_P \qquad (11)$$

and of the form

$$L = Q_{\mathrm{w}}^{+} Q_{\mathrm{w}} + V_P, \qquad (12)$$

where all coefficients of Q_{b} and Q_{w} are positive.

Interesting classes of such operators and their spectral theory in the space $\mathcal{L}_2(\mathbb{Z}^2)$ are studied in [1, 3]. In particular, the attention is paid to zero modes of the operator $Q_{\mathrm{b}}^{+} Q_{\mathrm{b}}$ (respectively, $Q_{\mathrm{w}}^{+} Q_{\mathrm{w}}$), which are functions $\psi \in \mathcal{L}_2(\mathbb{Z}^2)$ such that $Q_{\mathrm{b}} \psi = 0$ (respectively, $Q_{\mathrm{w}} \psi = 0$). Explicit solutions are found in the case of coefficients of the form $\exp(l_j(n))$, where $l_j(n) = l_{j1} n_1 + l_{j2} n_2$, $j = 1, 2$, are linear forms:

$$Q_{\mathrm{w}} = 1 + c e^{l_1(n)} t_1 + d e^{l_2(n)} t_2 = Q(c, d). \qquad (13)$$

Especially interesting is the case when $l_{ij} + l_{ji} = h$ is independent of i, j. In this case, the operators $Q(c, d)$ satisfy the relation

$$Q(c,d)^{+} Q(c,d) - 1 = q^2 (Q(c/q^2, d/q^2) Q(c/q^2, d/q^2)^{+} - 1), \qquad (14)$$

where $q = e^{l_{11}} = e^{l_{22}} = e^{(l_{12} + l_{21})/2}$. In some cases, these relations allow to establish a wonderful property of the spectrum of operators $L = Q^{+} Q$ and $L' = Q Q^{+}$ in the Hilbert space $\mathcal{L}_2(\mathbb{Z}^2)$ similar to that of the famous "Landau operator" in the magnetic field, though there is nothing like a physical magnetic field in our situation. This phenomena are coming only from the nature of the discrete systems.

A pair of operators $(Q_{\mathrm{b}}, Q_{\mathrm{w}})$ defines a discrete connection on the plane $\mathbb{R}^2$ as described in Section 1. As shown in [1], this connection has trivial local holonomy in the special case $l_{ij} + l_{ji} = 0$.

It is also noticed in [1] that operators $Q_{\mathrm{w}} = 1 + y_n t_1 + z_n t_2$ and $Q_{\mathrm{b}} = 1 + v_n t_1^{-1} + w_n t_n^{-1}$ define a connection of zero curvature if and only if there exists an everywhere non-zero function f such that

$$((Q_{\mathrm{w}} - 1)(Q_{\mathrm{b}} - 1) - 1) = f \cdot ((Q_{\mathrm{b}} - 1)(Q_{\mathrm{w}} - 1) - 1). \qquad (15)$$

4 Discrete analog of the operators ∂ and $\overline{\partial}$

Recall that we consider an equilateral triangular lattice $\ell \cong \mathbb{Z}^2$ in the Euclidean plane $\mathbb{R}^2$ and the corresponding b/w triangulation: the triangles $T_n^{\mathrm{w}} = \langle (n_1, n_2), (n_1+1, n_2), (n_1, n_2+1)\rangle$ are white, and the triangles $T_n^{\mathrm{b}} = \langle (n_1, n_2), (n_1-1, n_2), (n_1, n_2-1)\rangle$ are black, where $n = (n_1, n_2) \in \mathbb{Z}^2$.

We will consider the following operators on the space of functions on ℓ:

$$Q = 1 + t_1 + t_2, \qquad Q^+ = 1 + t_1^{-1} + t_2^{-1}, \tag{16}$$

which are formally conjugate to each other, and show that they have many properties similar in a sense to those of the operators $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial \overline{z}$ on the complex plane.

We denote by H the kernel of the operator Q^+. It will play a role of the space of *holomorphic functions.* It consists of functions ψ on ℓ that have zero sum over all the vertices of any black triangle: $Q^+\psi = 0$. We have not found any natural algebra structure on H, but we have constructed analogs of polynomials, the Taylor expansion, and the Cauchy formula. A discrete version of the maximum principle, which was considered in Section 2 in a more general situation, also takes place.

Now we introduce the space $\mathcal{P}_k$ of "polynomials of degree $\leqslant k$", which are solutions of the following system of equations:

$$Q^+\psi = 0, \qquad Q^{k+1}\psi = 0. \tag{17}$$

Proposition 3. *The dimension of the subspace $\mathcal{P}_k$ equals $2k+2$.*

Proof. By definition, $\mathcal{P}_0$ is the space of functions ψ satisfying both equations

$$Q\psi = 0, \qquad Q^+\psi = 0. \tag{18}$$

It means that ψ is a covariant constant with respect to the canonical connection of our triangulation. As we have seen in Section 1, in the case of trivial holonomy, the space of covariant constants is two-dimensional. Thus, we have

$$\dim \mathcal{P}_0 = 2.$$

Notice that, since the operators Q and Q^+ commute, the space H of "holomorphic" functions is invariant under Q: $Q(H) \subset H$. We claim that we also have $Q(H) \supset H$, *i.e.*, for any function $\varphi \in H$, the following system of equations is consistent:

$$Q\psi = \varphi, \quad Q^{+}\psi = 0. \tag{19}$$

Indeed, the system defines an *affine* discrete connection on the plane similar to the canonical connection from Section 1. One can easily check that the local holonomy of this connection is trivial if and only if we have $Q^{+}\varphi = 0$, which holds by assumption. The rest of the argument is the same as in the case of linear discrete connection.

Thus, the restriction of Q to the space H is a linear operator whose image is the whole space H and the kernel is two-dimensional. It follows immediately that

$$\dim(\mathcal{P}_{k+1}/\mathcal{P}_{k}) = 2.$$

□

Remark 3. One can show that, for any $\psi \in \mathcal{P}_k \backslash \mathcal{P}_{k-1}$, there exists a degree $k-1$ polynomial $u_{k-1}(z, \bar{z})$ with complex coefficients such that the following holds

$$\psi_n = \operatorname{Re}\left(\alpha e^{\frac{2\pi i}{3}(n_1 - n_2)}(z^k + u_{k-1}(z, \bar{z}))\right), \tag{20}$$

where $n = (n_1, n_2)$, $z = n_1 + e^{\frac{2\pi i}{3}} n_2$, $\alpha \in \mathbb{C}$.

Now we introduce an analog of the Taylor expansion for functions from H. We will need to consider "big black triangles" $T_n^{(k)} = \langle (n_1, n_2), (n_1 - 2k - 1, n_2), (n_1, n_2 - 2k - 1) \rangle$, where $n = (n_1, n_2) \in \ell$. A big black triangle is homothetic to an ordinary black triangle and has $2k + 2$ vertices of the lattice in each side.

We need the following preparatory lemma.

Lemma 4. *For any "holomorphic function" $\psi \in H$, any $n \in \ell$ and $k \geqslant 0$, there exists a unique "degree $\leqslant k$ polynomial" $p_k \in \mathcal{P}_k$ such that p_k coincides with ψ in $T_n^{(k)}$.*

Proof. We prove the existence by induction. For $k = 0$, the assertion is true, since there always exists a covariant constant p_0 coinciding with ψ in just one black triangle.

Assume that the assertion is proved for $k = l - 1$. Then we can find $p_{l-1} \in \mathcal{P}_{l-1}$ such that $p_{l-1} = Q\psi$ wherever in $T^{(l-1)}_{(n_1-1, n_2-1)}$. As we have already seen, we can find a function $\varphi \in H$ such that $Q\varphi = p_{l-1}$, and a covariant constant $p_0 \in \mathcal{P}_0$ such that $p_0 = \psi - \varphi$ in $T_n^{\mathrm{b}} = T_n^{(0)}$. We set $p_l = \varphi + p_0$. By construction, we will have: $p_l \in \mathcal{P}_l$, $p_l = \psi$ in $T_n^{(0)}$ and $Qp_l = Q\psi$ in $T^{(l-1)}_{(n_1-1,n_2-1)}$.

Thus, the difference $p_l - \psi$ satisfies the following

$$Q^+(p_l - \psi) = 0 \text{ everywhere}, \quad Q(p_l - \psi) = 0 \text{ in } T^{(l-1)}_{(n_1-1,n_2-1)},$$

which means that the sum of the values of the function $p_l - \psi$ over the vertices of any triangle, black or white, contained in $T_n^{(l)}$ is zero. Since $p_l = \psi$ in $T_n^{(0)}$, this implies that the function $p_l - \psi$ is identically zero in $T_n^{(l)}$.

The uniqueness of the "polynomial" p_k follows from the dimension argument: the dimension of $\mathcal{P}_k$ is $2k+2$ by Proposition 3, and the $2k+2$ values of ψ at vertices lying in one side of $T_n^{(k)}$, say, points $(n_1 - j, n_2)$, where $j = 0, 1, \ldots, 2k+1$, can be arbitrary as implied by the following lemma. $\square$

Lemma 5. *Let $Y_{n_1 n_2}$ be the following subset of the lattice ℓ:*

$$Y_n = \{(n_1, n_2)\} \cup \{(n_1 - j, n_2),\ (n_1, n_2 + j),\ (n_1 + j, n_2 - j)\}_{j \geqslant 1}, \qquad (21)$$

where $n = (n_1, n_2)$. Let V_n be the linear space of all real functions on Y_n. Then, for any $n \in \ell$ the restriction map $H \to V_n$ is an isomorphism.

In other words, in order to get a solution to the equation $Q^+\psi = 0$ one can set values of ψ in all points from Y_n arbitrarily, and then the equation will prescribe the values of ψ in all the other points.

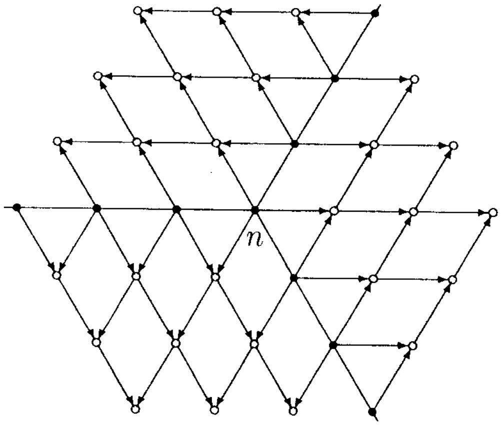
Proof. The three rays with the origin n at which all the points from Y_n lie, cut the plane into three parts, which are cyclically interchanged under rotation by $2\pi/3$ around n. Consider one of these parts $U = \{(n_1 - j_1, n_2 + j_2)\}_{j_1, j_2 > 0}$. All black triangles in U have the form $T^{\mathrm{b}}_{(n_1 - j_1 + 1, n_2 + j_2)}$, where $j_1, j_2 > 0$. The corresponding equations are:

$$\psi_{n_1 - j_1, n_2 + j_2} = -\psi_{n_1 - (j_1 - 1), n_2 + j_2} - \psi_{n_1 - (j_1 - 1), n_2 + (j_2 - 1)}. \qquad (22)$$

They allow to express recursively the value of ψ at any point from $\ell \cap U$ via the values of ψ at $\ell \cap \partial U$.

A similar situation takes place in the other two parts of the plane that are obtained from U by rotation by $2\pi/3$ around n. The recursion process is illustrated in Fig. 7, where the vertices from Y_n are marked by bold circles. $\square$

Figure 7: Recursive construction of ψ

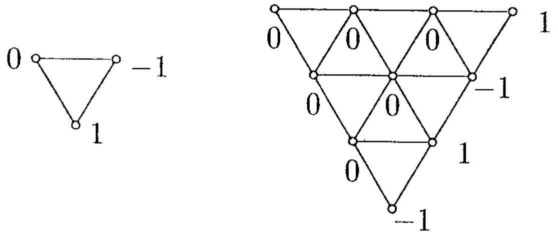


There is a canonical way to associate three "polynomials of degree k" $p_{k,1}, p_{k,2}, p_{k,3}$ with any big black triangle $T_n^{(k)}$ by setting them to be identically zero everywhere in $T_n^{(k)}$ except at vertices lying in one of the sides where they are equal to ± 1, namely:

$$\begin{aligned} p_{k,1}(n_1 - 2k - 1 + j, n_2) &= (-1)^{j+k}; \\ p_{k,2}(n_1, n_2 - j) &= (-1)^{j+k}; \\ p_{k,3}(n_1 - j, n_2 - 2k - 1 + j) &= (-1)^{j+k}, \end{aligned} \tag{23}$$

$j = 0, 1, \ldots, 2k+1$. In Fig. 8 the values of $p_{k,1}$ in $T_n^{(k)}$ are shown for $k = 1, 2$. The pictures for $p_{k,2}$ and $p_{k,3}$ are obtained by rotation by $2\pi/3$. Obviously,

Figure 8: "Polynomials" $p_{k,1}$ in $T_n^{(k)}$; $k = 1, 2$



we have

$$Qp_{k,i} = p_{k-1,i}, \tag{24}$$

where $p_{k,i}$ is associated with $T^{(k)}_{(n_1,n_2)}$, and $p_{k-1,i}$ with $T^{(k-1)}_{(n_1-1,n_2-1)}$.

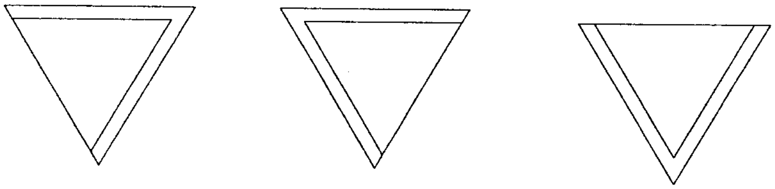
The polynomials $p_{k,j}$ are linearly dependent:

$$p_{k,1} + p_{k,2} + p_{k,3} \in \mathcal{P}_{k-1}, \tag{25}$$

where we assume that the polynomials are associated with the same big triangle $T^{(k)}_n$. Thus, in order to get a basis in the space of polynomials we have to select a big triangle $T^{(k)}_n$ for each k and a pair of polynomials $p_{k,i}, p_{k,j}$, $i, j \in \{1, 2, 3\}$. We do it as follows.

For a big triangle $T^{(k)}_{(n_1,n_2)}$, we define its *two-side extension* of type (12), (23), or (31) to be the big triangle $T^{(k+1)}_{(n_1+1,n_2+1)}$, $T^{(k+1)}_{(n_1+1,n_2)}$, or $T^{(k+1)}_{(n_1,n_2+1)}$, respectively (see Fig. 9).

Figure 9: Two-side extensions



type (12) type (23) type (31)

Let us fix a sequence of big triangles $T(0), T(1), \dots$ such that

1) $T(0)$ is an ordinary black triangle;
2) $T(k+1)$ is a two-side extension of $T(k)$ (thus, $T(k) = T^{(k)}_n$ for some $n \in \ell$);
3) the union $\bigcup_k T(k)$ is the whole plane $\mathbb{R}^2$.

Such a sequence will be called *admissible.*

With an admissible sequence of triangles we associate a basis ψ^1_j, ψ^2_j, $j = 0, 1, \dots$, in the space $\bigoplus_k \mathcal{P}_k$ of polynomials in the following way. If $T(k)$ is the two-side extension of $T(k-1)$ of type (ij), we set ψ^1_k and ψ^2_k to be the polynomials $p_{k,i}$ and $p_{k,j}$ associated with the triangle $T(k)$. We also denote by $T^{\mathrm{b}}(k)$ the (ordinary) black triangle $T^{\mathrm{b}}_{(n_1-k,n_2-k)}$, where $T(k) = T^{(k)}_{(n_1,n_2)}$.

By construction, we have $\psi^i_j = 0$ everywhere in $T(k)$ if $k < j$. Thus, any series of the form

$$\sum_{k=0}^{\infty} (\alpha^1_k \psi^1_k + \alpha^2_k \psi^2_k) \tag{26}$$

converges everywhere in ℓ, since, by the definition of admissible sequence, for any $n \in \ell$, we have $n \in T(k)$ for any sufficiently large k.

Theorem 4. *For any admissible sequence of triangles $T(0), T(1), \dots$ and any "holomorphic" function $\psi \in H$, there exists a unique series of coefficients $\alpha_0^1, \alpha_0^2, \alpha_1^1, \alpha_1^2, \alpha_2^1, \alpha_2^2, \dots$ such that the series (26) converges to the function ψ. Moreover, the coefficients α_k^1, α_k^2 can be found from the knowledge of the value of the* "kth derivative" *$Q^k\psi$ at the vertices of the triangle $T^{\mathrm{b}}(k)$.*

Proof. The first assertion of the theorem follows from the fact that we can approximate ψ in the triangle $T(k)$ by a "polynomial" of degree k and from $\lim\limits_{k\to\infty} T(k) = \mathbb{R}^2$.

It is straightforward to check that, if ψ is identically zero in $T(k)$, then $Q^k\psi$ is identically zero in $T^{\mathrm{b}}(k)$. Thus, $Q^j\psi_k^i$ is not identically zero in $T^{\mathrm{b}}(k)$ if and only if $j = k$. This implies the latter assertion of the theorem. ☐

Now we construct an analog of the Cauchy formula for recovering a solution ψ of the equation $Q^+\psi = 0$ in a domain from the knowledge of ψ at the boundary of the domain.

By a domain we will always mean a closed domain D in the plane such that D is a simplicial subcomplex of our regular triangulation. For a black or white triangle T, we will write $T \in \partial_+ D$ if we have $T \not\subset D$ and $T \cap \partial D \neq \varnothing$.

Consider the following function (the famous Pascal triangle, see Fig. 10).

$$G_{(n_1,n_2)} = \begin{cases} (-1)^{n_1+n_2}\dbinom{n_1}{n_1+n_2} & \text{if } n_1, n_2 \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

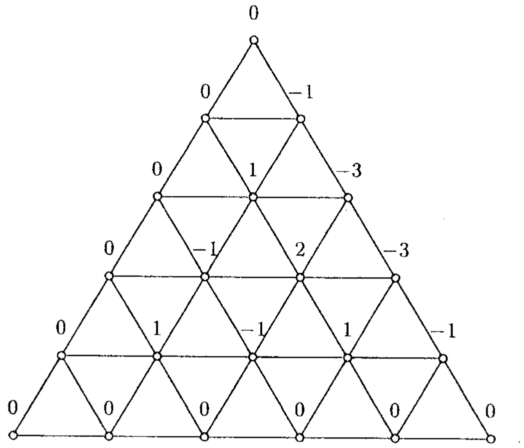
Lemma 6. *The function G is a fundamental solution to the black triangle equation, i.e., we have*

$$Q^+G = \delta, \tag{28}$$

where δ is the delta-function

$$\delta_n = \begin{cases} 1 & \textit{if } n = (0,0), \\ 0 & \textit{otherwise.} \end{cases}$$

Figure 10: The function G



Proof. The formula (27) can be rewritten in the form

$$G_n = \sum_{k=0}^{\infty} \left((-t_1^{-1} - t_2^{-1})^k \delta\right)_n . \tag{29}$$

Applying $Q^+ = (1 + t_1^{-1} + t_2^{-1})$ to both sides of (29) we obtain (28). $\square$

Proposition 4. *Let ψ be a "holomorphic function" in a finite domain D, i.e., a solution of the equation $Q^+\psi = 0$ in D. Then, for any $n \in D \cap \ell$, the following holds*

$$\psi_n = \sum_{T_m^{\mathrm{b}} \in \partial_+ D} (Q^+\psi)_m G_{n-m}, \tag{30}$$

where we set $\psi_m = 0$ for all $m \notin D$.

Note that the right hand side of (30) involves only the values of ψ at the boundary ∂D.

Proof. Notice that, if $T_m^{\mathrm{b}} \notin \partial_+ D$, then we have $(Q^+\psi)_m = 0$. Indeed, if $T_m^{\mathrm{b}} \subset D$, then the equality holds by assumption, and if $T_m^{\mathrm{b}} \cap D = \varnothing$, it holds, since ψ is assumed to be identically zero outside D. So, the right hand side of (30) can be written as

$$\sum_{m \in \ell} (Q^+\psi)_m G_{n-m}, \tag{31}$$

Denote this sum by $\widetilde{\psi}_n$.

We may assume without loss of generality that the domain D is contained in the sector $n_1, n_2 > 1$. Then, for any $n = (n_1, n_2)$ such that $T_n^{\mathrm{b}} \in \partial_+ D$, we have $n_1, n_2 > 0$. This implies

$$\widetilde{\psi}_n = \psi_n = 0 \quad \text{if } n_1 \leqslant 0 \text{ or } n_2 \leqslant 0. \tag{32}$$

From (28) we have

$$\begin{aligned}(Q^+(\psi - \widetilde{\psi}))_n &= (Q^+\psi)_n - \sum_{m\in\ell}(Q^+\psi)_m(Q^+G)_{n-m} \\ &= (Q^+\psi)_n - \sum_{m\in\ell}(Q^+\psi)_m\delta_{n-m} \\ &= (Q^+\psi)_n - (Q^+\psi)_n = 0.\end{aligned}$$

Thus, the function $\psi - \widetilde{\psi}$ belongs to H and, according to (32), vanishes at Y_0. It follows from Lemma 5 that $\psi - \widetilde{\psi} = 0$. □

In this proof, we have used two properties of the function G: the first one is relation (28), and the second one is that $G = 0$ outside the sector $n_1, n_2 \geqslant 0$. However, the latter is not needed as will follow from the next lemma.

Lemma 7. *For any function $\psi : \ell \to \mathbb{R}$ with finite support and any $\varphi \in H$ we have*

$$\sum_{m\in\ell}(Q^+\psi)_m\varphi_{n-m} = 0 \tag{33}$$

for all $n \in \ell$.

Proof. Denote by τ the following operator:

$$(\tau\psi)_n = \psi_{-n}.$$

We will have

$$\begin{aligned}\sum_{m\in\ell}(Q^+\psi)_m\varphi_{n-m} &= \sum_{m\in\ell}(Q^+\psi)_m(\tau\varphi)_{m-n} \\ &= \sum_{m\in\ell}\psi_m(Q\tau\varphi)_{m-n} \\ &= \sum_{m\in\ell}\psi_m(\tau Q^+\varphi)_{m-n} = 0.\end{aligned}$$

We used here the relation $\tau Q \tau = Q^+$ and the fact that the operators Q and Q^+ are formally conjugate to each other. ☐

Corollary 3. *The assertion of Proposition 4 takes place for an arbitrary function G satisfying (28).*

Appendix

Proof of Theorem 1. Any representation $\rho : \pi_1(M) \to GL(2, \mathbb{R})$ can be obtained by specifying a flat linear connection in the trivial two-dimensional vector bundle over the 1-skeleton of $\mathcal{T}$. This means that, to each oriented edge $\langle P, P' \rangle$, we associate a 2×2 matrix $R_{P,P'}$ so that, for any triangle $\langle P, P', P'' \rangle$ of $\mathcal{T}$, we have $R_{P,P''} = R_{P,P'} R_{P',P''}$ and $R_{P,P'} = R^{-1}_{P',P}$. For a fixed vertex P_0, a representation of $\pi_1(M, P_0)$ is obtained by putting $\rho(\gamma) = R_{P_0 P_1} \cdot \ldots \cdot R_{P_{m-1} P_m} R_{P_m P_0}$, where the path γ consists of the edges $\langle P_0, P_1 \rangle, \ldots, \langle P_{m-1}, P_m \rangle, \langle P_m, P_0 \rangle$. Clearly, any representation can be obtained in this way. Moreover, the representation is not changed under a "gauge transformation" $R_{P,P'} \mapsto C_P R_{P,P'} C^{-1}_{P'}$, where C_P, $P \neq P_0$, are arbitrary 2×2 matrices, and $C_{P_0} = \mathrm{id}$.

Let us fix a vertex P'_0 connected with P_0 by an edge of $\mathcal{T}$. By a gauge transformation, we can achieve $R_{P_0 P'_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let D be a simply connected domain containing P_0, P'_0. Consider a local covariantly constant section of the vector bundle, *i.e.*, a vector function $v_P = (v_{P1}, v_{P2})$ on vertices from D such that $v_P \cdot R_{P,P'} = v_{P'}$ for any edge $\langle P, P' \rangle \subset D$. Since we have $v_{P_0 2} = v_{P'_0 1}$, the section v_P can be recovered from the knowledge of $v_{P_0 1}, v_{P'_0 1}$. The space of covariantly constant local sections is two-dimensional.

By forgetting the second coordinate, we obtain a two-dimensional space of functions v_{P1}, which can be turned into the space of covariant constants of a flat discrete connection associated with $\mathcal{T}$. This can be done by putting

$$b_{T,P_1} = c_{23}, \quad b_{T,P_2} = c_{31} d_{23}, \quad b_{T,P_3} = c_{12} d_{21}, \tag{34}$$

where $T = \langle P_1, P_2, P_3 \rangle$ is a triangle of $\mathcal{T}$ with a fixed enumeration of vertices, $c_{ij} = (R_{P_i,P_j})_{21}$, $d_{ij} = \det(R_{P_i P_j})$. Changing the enumeration will result in the multiplication of all three coefficients (34) by the same constant, which has no effect on the corresponding discrete connection. Applying a generic gauge transformation with $C_{P_0} = \mathrm{id}$ we can always make all the coefficients (34) be non-zero.

The discrete connection $\{b_{T,P}\}$ obtained in this way has the following property. If we set $\psi_{P_0}, \psi_{P_0'}$ arbitrarily and extend ψ_P by solving equation (2) along a thick path, we will obtain the same function as if we set $v_{P_0} = (\psi_{P_0}, \psi_{P_0'})$, apply the parallel transport along the same path by using the connection R, and then forget the second coordinate of the obtained vector function. □

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