BLACK HOLE ATTRACTORS AND THE TOPOLOGICAL STRINGS

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WITH

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$$Z_{BH} = |Z_{top.}|^2$$

PLAN OF THE TALK

1. TOPOLOGICAL STRINGS (REVIEW)

2. EXACT BLACK HOLE ENTROPY (NEW)

3. INTERPRETATION

1. TOPOLOGICAL STRINGS

WITTEN 1988

WE WILL CONSIDER THE <u>A-MODEL</u>
FOR A CALABI-YAU 3-FOLD M.

- START WITH $\mathcal{N}=2$ SUPERCONFORMAL SIGMA-MODEL

$$\underset{\text{RIEMANN SURFACE}}{\Sigma} \longrightarrow \underset{\text{CALABI-YAU 3-FOLD}}{M}$$

- TOPOLOGICAL TWIST

FOR Σ : GENUS 0,

$$\langle \phi_A \phi_B \phi_C \rangle_{g=0} = C_{ABC}^{(\text{QUANTUM})}$$

$$= C_{ABC}^{(\text{CLASSICAL})} + \sum_{n} n_A n_B n_C N_{(n)}^{(0)} \frac{e^{2\pi i n\cdot t}}{1-e^{2\pi i n t}}$$

WHERE

- $C_{ABC}^{(\text{CLASSICAL})} = \int_M k_A \wedge k_B \wedge k_C$

$k_A \in H^{1,1}(M)$

- $t^A = \theta^A + i\gamma^A$

COMPLEXIFIED KÄHLER MODULI

- $n\cdot t = \sum_A n_A \cdot t^A$

QUANTUM COHOMOLOGY RING

THE MODULI SPACE OF THE A-MODEL IS SPECIAL KÄHLER.

KÄHLER : $G_{A\bar{B}} = \frac{\partial^2}{\partial t^A \partial \bar{t}^{\bar{B}}} K$

SPECIAL : INTRODUCE PROJECTIVE COORDINATES $X^0, X^1, \cdots, X^{h^{1,1}}$

$$t^A = \frac{X^A}{X^0}$$

$\exists$ HOLOMORPHIC PRE-POTENTIAL $F_0(X)$ SUCH THAT

- $F_0(\gamma X) = \gamma^2 F_0(X)$ HOMOGENEOUS
- $X^0 \frac{\partial^3 F_0}{\partial X^A \partial X^B \partial X^C} = C_{ABC}(t)$
- $e^{-K} = \mathrm{Im}\left(X^\Lambda \frac{\partial \bar{F}_0}{\partial \bar{X}^\Lambda} \right)$

IN THE B-MODEL FOR $\tilde{M}$ = MIRROR OF M.

$$X^\Lambda = \int_{\alpha_\Lambda} \Omega \ , \quad \frac{\partial F_0}{\partial X^\Lambda} = \int_{\beta^\Lambda} \Omega$$

Ω : HOLOMORPHIC 3-FORM

$\{ \alpha_\Lambda , \beta^\Lambda \}$ SYMPLECTIC CONJUGATE 3-CYCLES

$\Lambda = 0, 1, \cdots, h^{2,1}(\tilde{M}) = h^{1,1}(M)$

GENERALIZATION TO HIGHER GENUS RIEMANN SURFACES

A-MODEL CONTAINS $\mathcal{N}=2$ SUPERCONFORMAL CURRENTS :

$G_L^+ \, dz$, $G_R^+ \, d\bar{z}$ $\rightarrow$ TOPOLOGICAL BRST SYMMETRY

$G_L^- (dz)^2$, $G_R^- (d\bar{z})^2$ $\rightarrow$ "ANTI-GHOST"

ON GENUS g SURFACE Σ

$$\langle G_L^-(z_1) \cdots G_L^-(z_{3g-3}) \, G_R^-(\bar{w}_1) \cdots G_R^-(\bar{w}_{3g-3}) \rangle$$

MAKES A TOP FORM ON $\mathcal{M}_g$, MODULI SPACE OF Σ .

($\because$) η : BELTRAMI DIFFERENTIAL $\rightarrow \int_\Sigma \eta \cdot G_L^-$)

$$F_g = \int_{\mathcal{M}_g} \langle (G_L^-)^{3g-3} (G_R^-)^{3g-3} \rangle$$

THE TOPOLOGICAL STRING AMPLITUDE IS GIVEN BY SUMMING THIS OVER g :

$$F_{top}(t, \lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2} \, F_g(t)$$

λ : TOPOLOGICAL STRING COUPLING

$$F_1 = -\frac{2\pi i}{24} C_{2A} t^A$$

$$+ \sum_n N^{(1)}(n) \log\left(\frac{1}{\prod_{m=1}^{\infty} \left(1 - e^{2\pi i m n \cdot t}\right)} \right)$$

$$+ \frac{1}{12} \sum_n N^{(0)}(n) \log\left(\frac{1}{1 - e^{2\pi i n \cdot t}} \right)$$

$$F_g = \ldots$$

WE CAN COMPUTE F_g EXACTLY.

- HOLOMORPHIC ANOMALY EQUATION BCOV 1993

$$\frac{\partial}{\partial \bar{t}^{\bar{A}}} F_g = \frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}} e^{2K} G^{B\bar{B}} G^{C\bar{C}} \times \left(D_B D_C F_{g-1} + \sum_{r=1}^{g-1} D_B F_r D_C F_{g-r} \right)$$

- MIRROR SYMMETRY

  A-MODEL ON M = B-MODEL ON $\tilde{M}$

- STRING FIELD THEORY

  - QUANTUM KODAIRA-SPENCER THEORY (B-MODEL)
  - KÄHLER GRAVITY (A-MODEL)

    → QUANTUM FOAMS AND CRYSTAL MELTING
    OKOUNKOV, RESHETIKHIN & VAFA 2003

  FOR OPEN STRINGS:

  - CHERN-SIMONS GAUGE THEORY (A-MODEL)
    WITTEN 1992
  - MATRIX MODEL (B-MODEL)
    DIJKGRAAF & VAFA 2002

- DIRECT COMPUTATION OF $N^{(0)}(n)$, $N^{(1)}(n)$, ...

APPLICATION TO TYPE IIA SUPERSTRING ON $\mathbb{R}^4 \times CY_3$

(BCOV 1993)

KÄHLER MODULI X^Λ OF CY_3

$\rightarrow$ MASSLESS VECTOR MULTIPLETS IN $\mathbb{R}^4$

($\mathcal{N}=2$ SUPERSYMMETRY)

HOLOMORPHIC PREPOTENTIAL FOR LOW ENERGY THEORY

$$F(X, W) = \sum_{g=0}^{\infty} f_g(X)\, W^{2g}$$

W : WEYL SUPERMULTIPLET

$\ni$ (GRAVI-PHOTON FIELD STRENGTH)^2,

EULER DENSITY $R \wedge R$

$$f_g(X) \sim (X^0)^{2-2g}\, F_g^{top}(t)$$

$$t^A = \frac{X^A}{X^0}$$

THESE TERMS DO NOT DEPEND ON COMPLEX MODULI OF CY_3.

$\Longrightarrow$

MANY PHYSICAL APPLICATIONS

2. EXACT BLACK HOLE ENTROPY

TYPE IIA SUPERSTRING ON $\mathbb{R}^4 \times CY_3$

CONTAINS VARIOUS BLACK HOLE SOLUTIONS.

$\mathcal{N}=2$ SUPERSYMMETRIC BLACK HOLES

ARE CHARACTERIZED BY ELECTRIC/MAGNETIC CHARGES

WITH RESPECT TO GAUGE FIELDS

IN VECTOR MULTIPLETS X^Λ, $\Lambda = 0, 1, \cdots, h^{1.1}$

ELECTRIC CHARGES :

q_0 = # D_0 BRANES

q_A = # D_2 BRANES ON 2 CYCLE A IN CY_3

$A = 1, 2, \cdots, h^{1.1}$

MAGNETIC CHARGES :

p^A = # D_4 BRANES ON 4 CYCLE A IN CY_3

p^0 = # D_6 BRANES WRAPPING THE WHOLE CY_3

BLACK HOLE ATTRACTOR I

— CLASSICAL STORY (GENUS 0) —

FERRARA, KALLOSH & STROMINGER 1995

THE CLASSICAL EQUATIONS OF MOTION FIX THE KÄHLER MODULI OF CY_3 AT THE BLACK HOLE HORIZON BY

$$p^\Lambda = \mathrm{Re}\, X^\Lambda, \quad q_\Lambda = \mathrm{Re}\, \frac{\partial F_0}{\partial X^\Lambda}$$

$$\Lambda = 0, 1, \cdots, h^{1,1}$$

BY THE STANDARD BLACK HOLE THERMODYNAMICS, THE BEKENSTEIN-HAWKING ENTROPY IS GIVEN BY

$$S_{BH} = \frac{i}{2}\pi \left(q_\Lambda \bar{X}^\Lambda - p^\Lambda \frac{\partial \bar{F}_0}{\partial \bar{X}^\Lambda} \right)$$

$$= \frac{1}{4} \underbrace{e^{-K}}_{}$$

AREA OF THE HORIZON

EVALUATED AT THE ATTRACTOR VALUES.

THE ENTROPY DOES NOT DEPEND ON COMPLEX MODULI OF CY_3, WHICH MAY VARY CONTINUOUSLY.

BLACK HOLE ATTRACTOR II

— QUANTUM STORY (ALL GENERA) —

LOPES CARDOSO, deWITT & MOHAUPT
1998–99

$$F(X, W^2) = \sum_{g=0}^{\infty} f_g(X)\, W^{2g}$$

$$f_g(X) \sim (X^0)^{2-2g} F_g^{top.}\left(t^A = \frac{X^A}{X^0}\right)$$

AT THE BLACK HOLE HORIZON,

$$p^\Lambda = \mathrm{Re}\, X^\Lambda ,$$

$$q_\Lambda = \mathrm{Re}\, \frac{\partial F}{\partial X^\Lambda}, \qquad W^2 = 256$$

THE BLACK HOLE THERMODYNAMICS IN THE QUANTUM-CORRECTED LOW ENERGY THEORY GIVES

$$S_{BH} = \frac{\pi i}{2}\left(q_\Lambda \bar{X}^\Lambda - p^\Lambda \frac{\partial \bar{F}}{\partial \bar{X}^\Lambda}\right) - 256\pi\, \mathrm{Im}\left(\frac{\partial F}{\partial W^2}\right)$$

THE ENTROPY S_{BH} IS INDEPENDENT OF COMPLEX MODULI OF CY_3.

$$F(X, W^2 = 256) = -\frac{\pi}{2} i \, F^{top}(t, \lambda)$$

$$\left(t^A = \frac{X^A}{X^0} \, , \quad \lambda = \frac{4\pi}{X^0} \right)$$

THE BEKENSTEIN-HAWKING ENTROPY S_{BH} CAN BE CAST INTO A SIMPLER FORM BY INTRODUCING

$$\mathcal{F}(\phi, p) \equiv F^{top}(X) + \bar{F}^{top}(\bar{X})$$

$$X^\Lambda = p^\Lambda + i \frac{\phi^\Lambda}{\pi} .$$

WE FIND

$$S_{BH} = \mathcal{F} - \phi^\Lambda \frac{\partial}{\partial \phi^\Lambda} \mathcal{F}$$

AND THE ATTRACTOR EQUATIONS ARE

$$q_\Lambda = -\frac{\partial}{\partial \phi^\Lambda} \mathcal{F} .$$

$S_{BH}(q, p)$ IS THE LEGENDRE TRANSFORM OF $\mathcal{F}(\phi, p)$ FOR $q_\Lambda \to \phi^\Lambda$.

WE PROPOSE

$$\exp\left[F_{top.} + \bar{F}_{top.} \right] \quad \text{WITH} \quad X^{\Lambda} = p^{\Lambda} + i \frac{\phi^{\Lambda}}{\pi}$$

$$= \sum_{q} \Omega(q, p) \exp\left(- \phi^{\Lambda} q_{\Lambda} \right)$$

WHERE

$\Omega(q, p)$ = # OF MICROSCOPIC STATES
OF THE BLACK HOLE
WITH CHARGES (q, p).

$\ln \Omega(q, p)$: MICROSCOPIC ENTROPY

(NOTE :

$$S_{BH}(q, p) \sim \ln \Omega(q, p)$$

WHEN $|q| \gg 1$)

3. INTERPRETATION

$$Z_{BH} = |Z_{top}|^2$$

WHERE

$$Z_{BH} = \sum_q \Omega(q,p)\, e^{-\phi^\Lambda q_\Lambda}$$

$$Z_{top} = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right)$$

- SINCE THE COMPUTATION ASSUMES SUPERSYMMETRY, WE EXPECT THAT Z_{BH} IS A SUPERSYMMETRIC INDEX,

$$Z_{BH} = \mathrm{tr}\left[(-1)^F e^{-\phi^\Lambda q_\Lambda} e^{-\beta H}\right]$$

EVALUATED IN THE CFT DUAL OF THE NEAR-HORIZON GEOMETRY $AdS_2 \times S^2 \times CY_3$ OF THE BLACK HOLE.

$\Omega(q,p)$ = # BPS STATES WITH CHARGES (q,p), COUNTED WITH THE SIGN FACTOR $(-1)^F$.

- $Z_{top}(p + i\frac{\phi}{\pi}) = e^{F_{top}}$

HAS AN INTERPRETATION AS A <u>WAVE-FUNCTION</u>
ASSOCIATED TO QUANTIZATION
OF THE MODULI SPACE OF CY_3

(WITTEN 1993)

$$Z_{top}(p + i\frac{\phi}{\pi})\, \overline{Z}_{top}(p - i\frac{\phi}{\pi})$$

$$= \sum_q \Omega(q, p)\, e^{-\phi^\Lambda q_\Lambda}$$

$$\Downarrow$$

$\Omega(q, p) =$ WIGNER'S DISTRIBUTION FUNCTION

1932

$\Omega(q, p)$ CAN BE NEGATIVE

... OK SINCE IT IS THE INDEX OF BLACK HOLE

$(-1)^F$

NOTE:

$$\sum_q \Omega(q, p) = |Z_{top}(p)|^2 > 0$$

$$\sum_p \Omega(q, p) = |\tilde{Z}_{top}(q)|^2 > 0$$

$\Rightarrow$ DEFORMATION QUANTIZATION
OF THE MODULI SPACE OF CY_3.

OKOUNKOV, RESHETIKHIN & VAFA

IQBAL, OKOUNKOV, NEKRASOV & VAFA

MAULIK, NEKRASOV, OKOUNKOV & PANDHARIPANDE

2003

$$e^{F_{TOP}(t,\lambda)}$$

$$= \sum_{n} \rho(n)\, e^{-\lambda n_0 - t^A n_A}$$

FOR TORIC CY_3.

$\rho(n) = \#$ DIMER CONFIGURATIONS.

THIS IS RELATED TO COMPUTATION OF $\Omega(p,q)$ BY A CHAIN OF DUALITIES

$$\mathrm{IIA} \xrightarrow[T]{} \mathrm{IIB} \xrightarrow[S]{} \mathrm{IIB} \xrightarrow[T]{} \mathrm{IIA}$$

- THE INDEX $tr\,[(-1)^F e^{-\phi^\Lambda q_\Lambda} e^{-\beta H}]$

MAY DEPEND ON CONTINUOUS BACKGROUND MODULI AT THE ASYMPTOTIC INFINITY OF THE BLACK HOLE

DUE TO :
- AdS_2 FRAGMENTATION
- MULTIPLE BASINS OF ATTRACTION
- EXISTENCE OF NONCOMPACT DIRECTIONS

$\Updownarrow$?

HOLOMORPHIC ANOMALY IN TOPOLOGICAL STRING.

$$Z_{BH} = |Z_{TOP}|^2 = |Z_{CS}|^2$$

PHYSICS MATHEMATICS

SCIENTIFIC

HAPPY BIRTHDAY, ALBERT.

52ND

BEST WISHES FOR

MANY MORE ADVENTURES

IN MATHEMATICS & PHYSICS.